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NOTE

On Rewriting the History of the Foundations of Mathematics at the Turn of the Century

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Some of the best-known textbooks on the history of mathematics share what I call a "standard interpretation" of the origin and development of the set-theoretic paradoxes [Eves 1976, 473–483; Kline 1972, 1183–1210; among others]. One might describe the basic premises of this standard interpretation in the following way. Most scholars claim that Cesare Burali-Forti discovered the contradiction of the greatest ordinal number in 1897 [Bell 1945, 279; Bourbaki 1969, 49; Bunn 1980, 237; Clark 1975, 79–80; Copi 1958, 281; Hobson 1905, 170; Kattsoff 1948, 88; Kennedy 1963, 262; Kline 1972, 1003; Poincaré 1905, 822–823; among others]. Immediately after its publication, dozens of papers appeared dealing with the paradox [van Heijenoort 1967, 104] and, as a consequence, more paradoxes were encountered. It has been said that Georg Cantor came upon similar paradoxes connected with the greatest cardinal and ordinal numbers in 1899 [van Heijenoort 1967, 113; Dou 1970, 65]. However, recent studies assert that this discovery came, perhaps, as early as 1895 or 1896 and, therefore, that Cantor anticipated Burali-Forti in his discovery [Bell 1945, 55; Bochenski 1970, 388; Copilowish 1948, 64, note 2; Dauben 1979, 192; Dauben 1980, 212; Grattan-Guinness 1971, 365; Zlot 1957, 100]. According to the standard interpretation, Bertrand Russell presented another paradox in *The Principles of Mathematics* [1903], although he had discovered it in 1901, and described it in a letter to Gottlob Frege a year later [Russell 1944, 13; Russell 1956, 26; Russell 1959, 75–76; Grattan-Guinness 1978, 135].

Three main points should certainly be stressed in connection with this "standard" interpretation. First, it claims that paradoxes were originally encountered as the result of criticism of the theory of transfinite numbers [Kennedy 1970, 593]. Second, that discovery of the paradoxes made clear the need for a reexamination of the foundations of mathematics and, as a direct result, the paradoxes stimulated three major philosophical schools in mathematics [Struik, 1967, 161]. Finally, following the dichotomy proposed by Frank Ramsey [1926, 352–354], historians may generally assume that the semantic paradoxes were a direct product of the logical ones.

The grounds for this standard interpretation all seem very reasonable, and even make sense chronologically. Although there have been certain conflicts and dis-

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agreements, historians and researchers working in this area of study have not suggested solutions to the historiographical problems generated by the discrepancies [Garciadiego 1983, 1–34].

It was not until 1978, more than 80 years after its publication, that the fact that there was no paradox in any of Burali-Forti's papers of 1897, or in Cantor's letters to Richard Dedekind of 1899, was pointed out [Moore 1978, 308–309]. Nevertheless, a substantial number of authors were discussing the Burali-Forti paradox at the turn of the century. Consequently, the question arises: if Burali-Forti did not discover the paradox of the greatest ordinal number, then who did? There is no simple answer to this question [Moore & Garciadiego 1981, 331–342]. In fact, the so-called Burali-Forti paradox was the result of a slow process of metamorphosis in which several mathematicians and philosophers transformed the understanding of others. The first elements required for its formulation emerged in Russell's *The Principles of Mathematics*. Nevertheless, at the time, Russell thought he had resolved the difficulty. The argument familiar today took its present shape through the work of people like Philip Jourdain, Henri Poincaré, Louis Couturat, and others.

A detailed analysis of the origin of Burali-Forti's paradox has revealed its close relationship with the paradoxes of Cantor and Russell. In fact, it has recently been shown that Russell came upon all these paradoxes in May and June of 1901 [Garciadiego 1983, 148–184]. Furthermore, it has been demonstrated that Russell discovered the paradoxes as a consequence of his attempt to reconcile some of Cantor's results with Russell's belief in the real existence of the class (set) of all classes (sets), which Russell thought was contradictory. That is to say, Russell came upon the paradoxes once he had accepted Cantor's transfinite numbers, not as an outcome of his criticisms.

Now that a new interpretation of the origin of the paradoxes has been presented (including the semantical ones [Garciadiego 1985]), it is of fundamental importance to consider its implications. The traditional account of the effects of the discovery of the paradoxes should be scrutinized in all its details. For example, one must question the role of the paradoxes in the origin and development, among other issues, of the modern schools of thought in mathematics. Unfortunately, the standard interpretation has been used, even explicitly, to discourage mathematicians from becoming concerned with the analysis and study of the fundamental notions and concepts of mathematics [Sacks 1975, 523]. Nevertheless, the historical interpretation suggested here claims an equally logical and even more consistent alternative. This new interpretation presents a more interesting and fertile tool for the future analysis and history of the principles of mathematics. It will help convince scholars, including historians and mathematicians, to take another look at this important episode in modern foundations.

REFERENCES

Bell, E. T. 1945. *The development of mathematics*, 2nd. ed. New York: McGraw–Hill.

Bochenski, I. 1970. *A history of formal logic*, 2nd ed. New York: Chelsea.

Bourbaki, N. 1969. *Eléments d'histoire des mathématiques*. Paris: Hermann.

Bunn, R. 1980. Developments in the foundations of mathematics, 1870–1910. In *From calculus to set theory, 1630–1910*. I. Grattan-Guinness, ed., pp. 220–255. London: Duckworth.

Clark, R. 1975. *The life of Bertrand Russell*. London: Butler & Tanner.

Copilowish, I. 1948. *The logical paradoxes from 1897 to 1904*. Ph.D. dissertation. The University of Michigan.

Copi, I. 1958. The Burali-Forti paradox. *Philosophy of Science* **25**, 281–286.

Dauben, J. W. 1979. *Georg Cantor: His mathematics and philosophy of the infinite*. Cambridge, Mass.: Harvard Univ. Press.

——— 1980. The development of Cantorian set theory. In *From calculus to set theory, 1630–1910*, I. Grattan-Guinness, ed., pp. 181–219. London: Duckworth.

Dou, A. 1970. *Fundamentos de las Matemáticas*. Barcelona: Labor.

Eves, H. 1976. *An introduction to the history of mathematics*, 4th ed. New York: Holt, Rinehart & Winston.

Garciadiego, A. R. 1983. *Bertrand Russell and the origin of the set-theoretic paradoxes*. Ph.D. dissertation. University of Toronto.

——— 1985. The emergence of some of the non-logical paradoxes of the theory of sets, 1903–1908. *Historia Mathematica* **12**, 337–351.

Grattan-Guinness, I. 1971. Towards a biography of Georg Cantor. *Annals of Science* **27**, 345–391.

——— 1978. How Bertrand Russell discovered his paradox. *Historia Mathematica* **5**, 127–137.

——— (ed.). 1980. *From calculus to set theory, 1630–1910*. London: Duckworth.

Hobson, E. W. 1905. On the general theory of transfinite numbers and order types. *Proceedings of the London Mathematical Society, Second Series* **5**, 170–188.

Kattsoff, L. 1948. *A philosophy of mathematics*. Ames: Iowa State Univ. Press.

Kennedy, H. 1963. The mathematical philosophy of Giuseppe Peano. *Philosophy of Science* **30**, 262–266.

——— 1970. Cesare Burali-Forti. In *Dictionary of Scientific Biography*, C. C. Gillespie, ed., Vol II, pp. 593–594, New York: Scribner's.

Kline, M. 1972. *Mathematical thought from ancient to modern times*. New York: Oxford Univ. Press.

Moore, G. H. 1978. The origins of Zermelo's axiomatization of set theory. *Journal of Philosophical Logic* **7**, 307–329.

Moore, G. H., & Garciadiego, A. R. 1981. Burali-Forti's paradox: A reappraisal of its origins. *Historia Mathematica* **8**, 319–350.

Poincaré, H. 1905. Les mathématiques et la logique. *Revue de Métaphysique et de Morale* **13**, 815–835.

Ramsey, F. 1926. The foundations of mathematics. *Proceedings of the London Mathematical Society, Second Series* **25**, 338–384.

Russell, B. 1903. *The principles of mathematics*. London: Cambridge Univ. Press (1938, 2nd ed.).

——— 1944. My mental development. In *The philosophy of Bertrand Russell*, by P. A. Schilpp, pp. 3–20. Chicago: Northwestern University.

——— 1956. *Portraits from memory*. London: Allen & Unwin.

——— 1959. *My philosophical development*. London: Allen & Unwin.

Sacks, G. 1975. Remarks against foundational activity. *Historia Mathematica* **2**, 523–528.

Schilpp, P. A. 1944. *The philosophy of Bertrand Russell*. Chicago: Northwestern University.

Struik, D. J. 1967. *A concise history of mathematics*. New York: Dover.

Van Heijenoort, J. 1967. *From Frege to Gödel: A source book in mathematical logic, 1879–1931*. Cambridge, Mass.: Harvard Univ. Press.

Zlot, W. L. 1957. *The role of the axiom of choice in the development of the abstract theory of sets*. Ph.D. dissertation. Columbia University.