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In search of infinity

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N. Ya Vilenkin. *In Search of Infinity*. Boston. Birkhäuser 1995. Trad. Abe Shenitzer ISBN 3-7643-3819-9

The most interesting aspect of this book, yet another among dozens of recent popular accounts of the infinite, is the extent to which it incorporates names and results due to Russian authors over roughly the past century. Unfortunately, like the rest of the book, even on this score the information provided is scanty and the documentation nonexistent. Except for the curious high school student or inquisitive college undergraduate, this book offers little that has not been done better in other recent works along similar lines by Gordon Fraser, Eli Maor, Constance Reid, Rudy Rucker, Ian Stewart, and James Trefil, to name but a few of the best examples of this genre. Sadly, Vilenkin's book does not rank with any of these as an informative or even very interesting account of infinity in mathematics.

Some idea of how much is missing from this book may be gathered from a brief survey of its opening chapter (there are only four in all). *Infinity and the Universe* begins with accounts of creation from antiquity, emphasizing cosmologies in which the 'infinite' plays a prominent role in various ways. Here the Greek Presocratic philosopher Anaximander figures prominently, since he taught that the origins of all things were somehow related to what he called the 'ἄπειρον' or 'boundless'. There is no attempt, however, to place Anaximander's seemingly abstract ideas into the context of Ionic philosophy in general, nor in the dialectical tradition beginning with Thales and continuing through Anaximenes, who was apparently dissatisfied with 'the infinite' as too vague a concept to account satisfactorily for the world and argued instead in favor of 'air' as the universal first principle. It would have been helpful had Vilenkin considered the extent to which Anaximander's 'ἄπειρον' is really a mathematical concept at all, and what distinctions are to be drawn between the 'boundless' on the one hand and the true "infinite" on the other, especially if the latter is considered not metaphysically but mathematically. Unfortunately, there is no evidence that Anaximander ever considered any aspect of the mathematical infinite.

Discussion soon turns naturally to the famous Presocratic paradoxes of motion due to Zeno —which prompts Vilenkin to say that: "Zeno was the first to show that a segment can be decomposed into infinitely many parts each of which has nonzero length" [p. 7]. Exactly where Zeno proved this, or what he might have meant by such a statement, is never explained. Nevertheless, due to the contradictions that arose from assuming that continuity entailed infinite divisibility, Vilenkin notes Aristotle's opposition to Zeno and various attempts mathematicians made to avoid using the concept of infinity. He claims that Eudoxus, intent on denying the existence of infinitely small or infinitely large magnitudes, introduced his axiom that given any two magnitudes $n < m$, there must exist some integer

$$a > 0 \text{ such that } n{\cdot}a > m$$

(the so-called Archimedean property).

What Vilenkin fails to mention here is that the entire rationale behind the approach Eudoxus developed in his theory of proportion was spurred by the Pythagoreans' discovery of incommensurable magnitudes. Vilenkin never explains that the arithmetic material on which Books VII, VIII and IX of Euclid's *Elements* are based, reflects early Pythagorean arithmetic and a corresponding theory of ratio or proportion that was developed before discovery of incommensurable magnitudes. Consequently, these books were *not* sufficiently rigorous to accommodate the discovery of incommensurable magnitudes, and it was the burden of Book V of the *Elements*, representing Eudoxus' extraordinary contribution to the infinite in mathematics, which reflected his new theory of proportion. This theory was explicitly designed to accommodate incommensurable magnitudes, and this in turn depended on the rigorous definition Eudoxus gave of what is now called the 'Archimedean' property of geometric magnitudes, whether commensurable or incommensurable (see, for example, studies by [Knorr 1975] and [Fowler 1987], as well as [Dauben 1984], for details).

Regrettably, none of this is ever mentioned, including the ancient Greek method of anthyphairesis, which is also connected with the discovery of incommensurable magnitudes, which likewise involved the infinite and would have been a natural, even provocative subject for Vilenkin's book. Nor is π or the remarkable fact that its approximation in both ancient Chinese and Greek mathematics involved infinity, specifically in the form of approximations derived by methods of exhaustion pioneered by Eudoxus and applied by Archimedes in the West, as well as by Liu Hui and his successors, independently, in the East (the contributions of Chinese mathematicians to the evaluation of π are recounted in [Li and Du 1987], and in [Martzloff 1988]).

One of the more remarkable statements Vilenkin makes in the first chapter of this book concerns Hypatia:

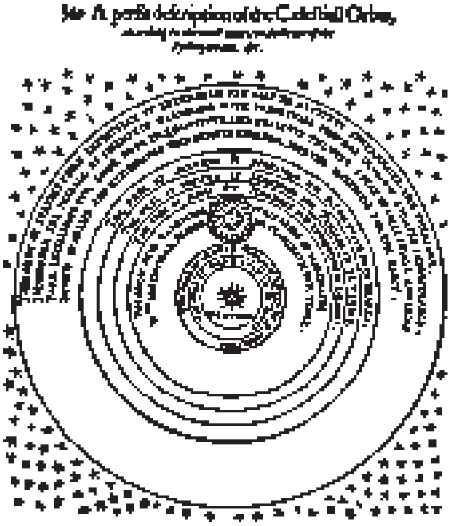
> In 415 AD a mob of Christian fanatics, incited by the Alexandrian bishop Cyril, lynched Hypatia, one of the last representatives of ancient culture, and burned the Alexandrian library that preserved the treasures of that culture [p. 10].

Thereafter, philosophy, says the author, 'was reduced to the role of a servant of theology', a conclusion that overlooks at least another hundred years or so of important intellectual activity, including the extraordinary work of Johannes Philoponus (for basic appreciations of Philoponus, there are numerous recent studies by Richard Sorabji, including [Sorabji 1987], and [Wildberg 1987]), not to mention the contributions of Syrianus, Proclus, Ammonius, and Simplicius (for references see [Sorabji 1988 and 1990]). As Aria Dzielska remarks in her authoritative biography of Hypatia, "we cannot maintain that Hypatia's death marked the demise of Alexandrian science and philosophy". And as if in direct contradiction of Vilenkin's position, she goes on to say that Hypatia was *not* "the last of the Hellenes". [Dzielska 1995, 105]

Although Vilenkin makes a great point of emphasizing Hypatia, he never mentions that she was the daughter of Theon of Alexandria, who was himself an accomplished mathematician. What Hypatia had to do with the problem of infinity is never explained, nor does Vilenkin take the opportunity to investigate infinity as it was later treated with such insight and innovation by Philoponus (as Sorabji shows in his study of Philoponus, in *Infinity and the Creation* [Sorabji 1987, 164-178].

From Hypatia and the Decline of the Roman Empire, Vilenkin's discussion suddenly jumps to the 16th century, overlooking entirely contributions to the subject of the infinite made in China, India and the Islamic world generally. Instead, Vilenkin's interest focuses on the infinities of space and time as conceived as a result of the Scientific

Revolution. Although Giordano Bruno is mentioned briefly [pp. 11-12], the first to actually depict the boundless nature of space, a natural consequence of the Copernican system, is not even mentioned! Although the accompanying illustration, from Thomas Digges' *A Perfit Description of the Caelestiall Orbes* [1576], only came to wide scholarly attention in 1934, and was later popularized in the well-known works of the French historian of science, Alexandre Koyré, none of this, unfortunately, is cited by Vilenkin. Considering the topic of Vilenkin's book, it is surprising that no mention is made of what is perhaps Koyré's best-known work, *From the Closed World to the Infinite Universe* [Koyré 1957]. This book, in fact, has done a great deal to introduce generations of students to the role infinity played in Renaissance cosmology.



Thomas Digges's diagram of the infinite Copernican universe (from *A Perfit Description of the Caelestiall Orbes*, 1576)

A page later, when Vilenkin turns to the 17th century, he mentions briefly Newton and Leibniz (although, again surprisingly, he makes no mention of Newton's controversial notions of infinite absolute space). On the other hand, the calculus and its use of 'infinitely small and infinitely large magnitudes', is dispatched within two sentences, with no real analysis or description at all!

Among the French, although Vilenkin mentions Fontenelle's *On the Multiplicity of Worlds* [p. 13], he says nothing of the role infinity may have played in this work. Even more remarkably, for a book about the mathematical infinite, there is no mention of L'Hospital, whose great contribution to the infinite in mathematics, especially in the form of the calculus and infinitesimals, was the first textbook ever on the subject, *Analyse des infiniment petits* [1797] (and hence the 'analysis' of modern mathematics).

On page 14 the author makes the remarkable assertion that "as a result of the successes of Newtonian mechanics and astronomy, the world picture proposed by Newton gained universal acceptance", as if the objections directed against Newton by the likes of Leibniz, Bentley, and the Bernoulli's were never raised or of no consequence! From 'Newton's World' the rest of this chapter jumps to a consideration of non-Euclidean geometry and the curvature of space, which leads Vilenkin to the question of whether the universe is infinite or not. In discussing the related question of the nature of matter and whether or not it is to be taken as truly infinitely-divisible [p. 31], the author notes vaguely that the neutron was discovered in the 1930s (why not say it was Chadwick who did so in 1932?)

He also says that at this same time, 'the neutron-positron model of the atomic nucleus came into being'. (Presumably here the author/translator meant the 'neutron-proton' model of the nucleus. A positron is a positive electron, first discovered as a result of consequences following from Dirac's mathematical wave model of the electron, which led the way to the concept of 'antimatter'). On the other hand, it was the neutron that proved essential, for as Vilenkin notes, "this model was so apt that it made possible the solution of the problem of the release of atomic energy" [p. 31]. But in all fairness it should be said that the crucial, initial insight had actually been made decades earlier when Albert Einstein first showed the equivalence of matter and energy in terms of his famous equation, derived in 1905, equivalent to the well-known $E = mc^2$. And for accuracy's sake, it should be made clear that what really inspired the model that led to the release of atomic energy was discovery of the neutron (as Leo Szilard realized immediately), for it was the neutron that made it possible, at least theoretically, to split the atom and trigger a tremendous release of energy through a series of chain reactions.

Chapter 2 of Vilenkin's book is concerned more with mathematics than physics, and deals with 'the mysteries of infinite sets'. There is a long, overly belabored discussion of 'The extraordinary hotel, or the thousand and first journey of Ion the Quiet'. What Vilenkin presents rather obscurely here, and better known as 'Hilbert's' hotel, is all explained more succinctly and more clearly, for example, in Rudy Rucker's *Infinity and the Mind* [1982, 73-75]. At the end of this chapter, Vilenkin introduces transfinite numbers and their corresponding transfinite arithmetic, but the material on transfinite order types is never linked with the theory of transfinite cardinal numbers. A greater surprise is the fact that the most important of Cantor's results, his conjecture —better-known as the Continuum Hypothesis— that among transfinite cardinal numbers, the power of the set of all real numbers $2^{\aleph_0} = \aleph_1$, is not introduced here, and is only mentioned briefly in Chapter 4. As for the lucidity of the material that Vilenkin presents on transfinite numbers, instead of presenting transfinite ordinal numbers in Chapter 2 as 'labels' for certain infinite sets, it would have been much clearer had the author simply focused on the order types of well-ordered sets, as Cantor himself did.

Chapter 3, 'Remarkable Functions and Curves, or a Stroll Through a Mathematical Hall of Wonders', begins with a brief discussion of the history of the concept of function. The author then moves on to introduce Jordan curves, and the famous Peano curve, or space-filling line, which eventually leads the author to a brief discussion of the history of measure theory, and of course to the great insights of Lebesgue and his definition of the Lebesgue integral.

In the Soviet Union a group of young mathematicians, mostly the students of N. N. Luzin who was the founder of the Russian school of the theory of functions and an enthusiastic proponent of set theory, firmly asserted that "Our god is Lebesgue. Our idol the integral" [p. 107]. This group, the *Luzitania* as they came to be known, provides the most interesting material in Vilenkin's book. Beginning with a description of the St. Petersburg Academy of Science in the 19th century, associated with such figures as Ostrogradski, Chebyshev, Markov, Lyapunov, Steklov and Korkin, the emphasis was on analysis and concrete mathematical problems; in St. Petersburg, the new set theory and discontinuous functions were not of much interest.

In Moscow, however, by the beginning of the 20th century courses were offered in set theory at Moscow University, where I. I. Zhegalkin presented his master's thesis on the subject of transfinite numbers in 1907. Others there who became interested in set theory included Egorov and Luzin, but unfortunately, rather than provide details and useful historical insights, Vilenkin leaves his readers bewildered with little more than vague generalities. "We have mentioned only some of Luzin's many students. Many of them have achieved international fame". But this is all that Vilenkin says, and neither these presumably important figures, nor what they accomplished, are elaborated in any further detail [p. 109].

The closing chapter of the book, 'In search of the absolute', takes up the well-known subject of the paradoxes of set theory. Vilenkin considers the various approaches taken to exclude the paradoxes from mathematics, and to retain the 'paradise' Hilbert proclaimed transfinite set theory to have created for mathematics. Remarkably, this complex story is told in little more than fifteen pages, and is understandably unsatisfying considering the detailed and substantial books that have appeared recently on this subject, especially Gregory Moore's *Zermelo's Axiom of Choice. Its Origins, Development, and Influence* [Moore 1980] and Alejandro Garciadiego's *Bertrand Russell and the Origins of the Set-theoretic Paradoxes* [Garciadiego 1992]. Vilenkin and/or his translator, Abe Shenitzer, should have at least mentioned these important works, if only to give readers (who may want to learn a great deal more than this book conveys) some idea of where best to look for more satisfying treatments of the infinite.

Vilenkin brings his *search for infinity* to a close with a discussion of intuitionism, which the author admits failed to convert most mathematicians, whereupon he adds the following closing remark:

> We note that in the last few decades interest in intuitionism has again been on the increase and many eminent logicians have explicitly or implicitly joined this trend in mathematical thought.

But who are these eminent logicians? Why has interest in intuitionism suddenly awakened? And is this the case in East and West alike? Vilenkin might have discussed counterparts in the Soviet Union to Erret Bishop, for example, but Vilenkin does not even mention Bishop, and the closing chapter is curiously silent on Russian contributions to the debates over foundations. Whether they may have favored formalism or intuitionism, or some form of neoplatonism, readers will not find answers in this book.

Indeed, if readers of this book are truly *In Search of Infinity*, they will be disappointed to find that the search is by no means over. This book hardly scratches the surface of the subject, and where it might have provided substantial insights to the subject as it was treated in the Soviet Union over the past century, what it offers instead is only a pale reflection, dropping a few prominent names and telling stories that are basically already known to the well-read student.

The three-page conclusion is especially vague, and at the end of the author's entire 'search', we are left with platitudes: "This process (the search for infinity) will never end, for the idea of the infinite is inexhaustible and the human mind will forever find in it new aspects" [p. 136]. Unfortunately, a reader's patience is not inexhaustible, and the few 'new aspects' concerning the infinite to be found in this book hardly warrant its having been translated into English. As a final confirmation that neither the author nor the translator were prepared to make the book really useful to readers, no attempt has been made to provide an index, and there is no bibliography. To add insult to injury, no citations of sources are provided, and no credit is given to the works of others upon which the author presumably relied in writing this book. Without an index of either names or subjects, it is virtually impossible to use the book as any sort of ready reference work.

Ultimately, there is nothing to recommend this book over the others on infinity noted at the beginning of this review. One can only wonder why the translator decided to undertake this project, or why he did not choose a more substantial work to translate, if indeed the choice was his.

Joseph W. Dauben, ex-editor de *Historia Mathematica* y ex-presidente de la Comisión Internacional de Historiadores de las Matemáticas, ha escrito ensayos sobre Georg Cantor, Charles S. Peirce, Abraham Robinson y sobre bibliografía e historiografía de las matemáticas, entre otros temas. Su *Georg Cantor. His mathematics and his philosophy of the infinite* (Cambridge University Press (1979) y Princeton University Press (1990)), es una detallada biografía intelectual del fundador de la teoría de los números cardinales y ordinales transfinitos. Recientemente publicó una extensa biografía sobre Abraham Robinson [*Abraham Robinson. The creation of nonstandard analysis*. Princeton University Press, 1995]. Actualmente labora en el Departamento de Historia de la Universidad de la Ciudad de Nueva York (CUNY) y sus intereses académicos se concentran en el estudio de las matemáticas orientales (especialmente las chinas).

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