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Labyrinth of Thought

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José Ferreirós Domínguez, *Labyrinth of Thought. A History of Set Theory and its Role in Modern Mathematics*. Basel: Birkhäuser, 2000. Pp. xxi + 440. ISBN 0817657495

Labyrinth of Thought constitutes a considerable expansion of an earlier Spanish language book (Ferreirós 1996) that is in turn an expansion and revision of a doctoral dissertation and that also incorporates material from several recent papers. Clearly, Ferreirós has worked extensively on this material and has thoroughly mastered a vast body of primary and secondary sources. With this book, Ferreirós' work on the history of set theory becomes readily accessible in a single volume so that Ferreirós' many interesting, though probably controversial, revisions to the standard histories of set theory can be widely disseminated, discussed, and evaluated. The overall quality of the writing and of the production is excellent and the name and subject indexes and the substantial bibliography are useful and informative.

Ferreirós stresses that he has taken a collective approach to the history of set theory, rather than follow the biographic approach of focussing on a single author, especially Cantor, that has been more common in works on this subject. Although he is quite aware of the difficulties of his collective approach to the material, it does make Ferreirós' book very ambitious, covering a great amount of material and a long timeframe. Overall the work is extremely successful, though the first two sections are much stronger than the third, which tends to be somewhat encyclopedic. The book is very useful in that it synthesizes many strands in the history of mathematics where sets played a role, analyzing the emergence of set theory in the context of the modernization of mathematics in the nineteenth century and its role in the work on the foundation of mathematics in the twentieth. Since Ferreirós seems to aim at a middle level of technical sophistication, the work will be accessible (though difficult at times) to a novice reader, but there is much material that will be of interest to specialists as well. Ferreirós provides an excellent summary of the work on Cantor, Dedekind, the paradoxes, and the foundations of mathematics, and offers the clearest summary of Riemann's difficult *Habilitationsvortrag* of 1854 that I have ever read (p. 61). Although this book must be classified as history of mathematics, philosophical readers will also find much material of interest. In one section Ferreirós apologizes for the philosophical issues that he raises and invites readers to skip ahead (p. 120), a novel approach to increasing reader interest in the philosophical parts of the book, since few could resist the temptation to see what they are missing.

For analytical purposes, Ferreirós distinguishes between the development of set theory as a separate branch of mathematical research, the use of set theory as a basic tool or language of mathematics, and the use of set theory as a foundation for mathematics, finding a complex interaction between these three ways in which set theory developed in his story of the development of modern mathematics and the work on foundations. Ferreirós discovered that the use of sets as a foundation of mathematics can be found surprisingly early in the work of Dedekind and that the use of sets as a basic tool or language of mathematics dates from the work of Riemann or even Gauss. Therefore, both of these developments predate Cantor's development of set theory as an independent branch of mathematics, contrary to the standard historical account. Dedekind emerges as a true precursor not only of set theory, as can be seen in his famous definition of infinite set which predates Cantor's work, but also of later work on foundations in Ferreirós account. Cantor fans will be pleased to know that he is still given full credit for the idea of the transfinite numbers and the establishment of set theory as an independent branch of mathematics (formalized by Zermelo) and that Cantor is given the largest photograph on the cover of the book.

Ferreirós argues that Riemann's notion of a 'manifold' should be seen not only as a concept in differential geometry but also with more generality as a 'set' when Riemann's work is considered as the development of a very abstract mathematics. Not many mathematicians were able to make use of the abstract generality of Riemann's work, however, until Dedekind and Cantor successfully applied it to point set topology. Ferreirós demonstrates the influence of Riemann on Dedekind and delineates the fundamental role of sets in their work, noting that Dedekind's concepts in field theory are taken up by Cantor and used in a set-theoretical fashion. Ferreirós' exploration of the use of the concept of set in mathematics prior to the work of Cantor provides a background to Cantor's early work, but it is somewhat unclear what the early use of sets in mathematics explains. As the subtitle announces, Ferreirós' book is a history of set theory and its role in modern mathematics, not a history of the concept of set and its role in modern mathematics. There clearly can be no role for set theory (as a separate field) prior to its founding by Cantor and Zermelo. Indeed, as Ferreirós emphasizes, it is only with the rise of model theory in the 1950s that set theory came to be seen as the natural axiomatic framework for mathematics (p. 338). The concept of set, however, can be traced back much further, as Ferreirós shows, but it is not clear why this fact should be of much interest, since the earlier uses predate the startling discoveries of Cantor about the transfinite and the idea that sets are a natural foundation for all of mathematics. In other words, all of the interesting features of set theory are missing from these early uses of the concept of set. What Ferreirós is providing, however, is a much longer view of the history of sets in mathematics, the context of Cantor's work, and a description of several aspects of the development of mathematics, not just work on foundations.

Ferreirós does an excellent job of sorting out the complex relationship between Cantor and Dedekind and provides some new information and plausible arguments about when they met, what information they exchanged, and why they broke off contact. He even has an explanation as to why Cantor did not give Dedekind the citations that were due to him, suggesting that early in his career Cantor could not afford to strain his good relations with the Berlin mathematicians who were at odds with the Göttingen mathematicians with whom Dedekind was associated. Ferreirós also deals judiciously and fairly with the delicate issue of Cantor's mental illness and long (though temporary) withdrawal from mathematics. Ferreirós tries to correct the impression that acceptance of the actual infinite is unique to Cantor by pointing out that the concept is fairly widely accepted among German mathematicians before Cantor's work appeared, notably by Bolzano and Dedekind. Cantor did, of course, discover the infinite hierarchy of ever larger infinite sets and first conceive of the transfinite sets as numbers. As Ferreirós explains, Cantor used the concept of well-ordering to define addition and multiplication on the transfinite numbers, thus justifying the application of the term 'number' to them (p. 275). Although Ferreirós gives an excellent critique of Cantor's arguments for the existence of the transfinite, he does not have much to say about the thorny issue of why the transfinite sets can be taken as numbers, beyond giving

Cantor's view. Ferreirós seems a little hasty when he argues that one is begging the question by asking that the transfinite share all properties with the natural numbers (pp. 269-71) —the issue is what properties are to be taken as essential and how you can know which properties are essential to numbers. How did Cantor decide on well-ordering as the essence of number?

It has been widely noted that the set-theoretical paradoxes were known to several German mathematicians, including Cantor, Bernstein, Hilbert and Zermelo, before Russell announced his famous paradox to Frege and published it in 1903. Ferreirós shows that Cantor was not shocked by the paradoxes because he distinguished between the absolute and the transfinite infinite and argued that the paradox of the universal set shows that the absolute cannot be determined (pp. 261-2; 291). On the basis of Ferreirós' research, the longstanding mystery of why only Russell recognized the paradoxes as such might be explained by the fact that only Russell saw the paradoxes as a refutation of the principle of comprehension. I suggest that mathematicians did not see, as Russell did, that the principle of comprehension led to a contradiction and needed to be replaced or amended; they applied the results much more narrowly to merely exclude certain large sets. Russell's diagnosis of the problem, no doubt helped tremendously by Frege's explicit formulation of the principle of comprehension, may provide the key to seeing the importance of the paradoxes. Ferreirós highlights the often-overlooked fact that there were two radically different kinds of solutions to the paradoxes, Russell's theory of types and Zermelo's axioms, both of which were extended by others, noting that this led to a bifurcation of later work on set theory. This bifurcation is also prefigured in the differing attitudes of Dedekind and Cantor at the beginning of their research, since only Dedekind saw set theory as a foundation of mathematics, while in his early work, Cantor saw set theory as a rigorous development of Riemann's ideas about manifolds (p. 202). While Bolzano and Dedekind wanted to prove the existence of infinite sets in order to use sets as a foundation of mathematics, Zermelo explicitly gave this up and simply took the existence of infinite sets for granted in order to develop his axiomatization of set theory (p. 234). It would be interesting to know whether this bifurcation would cleanly distinguish philosophical and mathematical approaches to set theory, especially given the unusual and high level of collaboration between mathematicians and philosophers during the early twentieth century. According to Ferreirós, Church emphasized the similarity between type theoretical approaches to set theory and Zermelo's axiomatic approach.

leading to a reunification of the two approaches in the work of Ramsey, Gödel and Tarski, all of whom are considered both philosophers and mathematicians.

One of the most provocative parts of Ferreirós book labels Dedekind a logicist, and even more important as a logicist than Frege, since Frege's works were widely ignored (p. 253, he notes that Schröder and Peirce associate logicism with Dedekind rather than Frege). Basically, Ferreirós argues that Dedekind gives sets and mappings a fundamental role in mathematics (numbers can be defined as sets, etc.) and that Dedekind considered sets to be part of logic, rather than part of mathematics proper. Sets are part of logic, according to Dedekind, since the principle of comprehension shows that there is a set corresponding to every concept and concepts are the elements of logic. Therefore, Ferreirós can claim that Dedekind is a logicist in the sense that if mathematics can be reduced to properties of sets, and if sets are part of logic, then mathematics can be reduced to logic (p. 226). This argument is convincing, as far as it goes, but Dedekind's conception of logic is problematic. Traditional logic is completely inadequate to serve as a foundation of mathematics, so it is not clear that saying that Dedekind would found mathematics on logic actually says anything more than simply that Dedekind would found mathematics on sets. Was it really important to Dedekind that mathematics be shown to be part of logic rather than a separate domain that could possibly have a special epistemological status? That Ferreirós calls Dedekind a logicist becomes even more puzzling when Ferreirós calls attention to Dedekind's generally Kantian epistemology (p. 241), if we consider that logicism was developed as a philosophical position in direct opposition to Kant's view of mathematics, so no one can be both a Kantian and a logicist. Indeed, Ferreirós himself provides a quote from Dedekind that seems in direct opposition to Kant (p. 243). Since Ferreirós admits that we do not have enough evidence to reconstruct Dedekind's philosophical views (p. 242-3, footnote), all claims in this area may be suspect, but one way to make Ferreirós' claim consistent is to read Dedekind as a neo-Kantian, rather than as a Kantian. In the nineteenth century, neo-Kantians gave up Kant's intuition as a form of sensibility but left the concepts of the understanding in place. Mathematics lost both its association with space and time and its special status as intuitive knowledge, being seen instead as purely conceptual and logical by the neo-Kantians. As Ferreirós points out, Dedekind explicitly argues that number concepts should be independent of any intuition of space or time in order to distance himself from Hamilton's

Kantian view (p. 221). Dedekind's rejection of Kant's intuitive foundation of mathematics is consistent with later neo-Kantian views, but not Kant himself, so interpreting Dedekind as a neo-Kantian could thus explain both the Kantian elements that Ferreirós finds in Dedekind's work and his explicit rejection of the Kantian view of mathematics.

How much is gained by the collective approach over and above the contributions of Ferreirós to our understanding of particular issues? It is certainly welcome to have an overview of the development of set theory and its relation to the foundations of mathematics in a single work and Ferreirós has also provided an excellent account of the relation of the mathematical projects of Dedekind and Cantor and an account of the role of sets in mathematics, a big topic that has been neglected in the past. *Labyrinth of Thought* is clearly an extremely important contribution to the history and philosophy of mathematics and a work that should become one of the major studies of set theory, to be read and discussed for years to come.

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