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Elucidating through history. The case of a well ordered set.[1]

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ABSTRACT

The goal of this talk it to explicitly show how the history of mathematics helps illuminate the pedagogic process. The assertion is even stronger. I attempt to exemplify how the history of mathematics might be used to elucidate the origins of certain mathematical concepts. I use the vocable ‘elucidating’ with a deeper connotation meaning that, because of the poor and disappointing results in the

1. This is simply an abstract of a much longer and detailed version of an essay that will be submitted to a new electronic journal of the *Mathematical Association of America* (edited by Dr. Victor Katz) on the relations between the history and pedagogy of mathematics. During the personal presentation at the conference, I will resort to specific examples of *elementary* mathematics to clarify my goal.

teaching of mathematics, every professor must approach his lecturing "as an attempt to throw light in some way on a [very] complex subject"; and not merely, "as to make the nature or meaning of something clear".[2] Thus, I do not simply mean that the teacher should complement the traditional lecturing techniques with elements that might enrich, embellish or simplify explanations. We should resort to history to find a more precise and deeper understanding of how some mathematical concepts emerged.

By the word 'history', I do not simply understand the passive narration of past events, but a critical and analytical process of reconstruction of ideas. Non professionals resort to history in their attempts to find specific data (*e.g.,* names, places, dates, among others), to entertain students, by enriching or embellishing the lecture. A listener to a popular lecture once commented that one might bend or distort history, if the new accounts clarified student's understanding. Any historian will reject this practice immediately. Similarly, one may request a mathematician to change the original conditions of a theorem to make it more appealing.

We should not confuse 'history' with 'chronology' and satisfy ourselves by learning who first proposed an idea and where and when he did it. The basic issue in this context is to explain *why* the concept (method or technique) was required in the first place.

2. The American Heritage Dictionary. Boston: Houghton Mifflin Co. 1982. (2nd College Edition). Page 477.

On the other hand, as Plato argues,[3] there are different levels of knowledge (imagination, belief, mathematical and ideal forms), and we may argue too that there are also different levels of comprehension to our students, of course not necessarily Platonist. To make a more profound impression on students some professors include extra mathematical examples of our daily life and some others narrate colloquial anecdotes of those associated with the topic under discussion. Some of these statements (and learning) might belong to the two first Platonist levels of knowledge: those of fantasy and opinion. But, I argue, if we desire to provide a deeper or more meaningful understanding to our students we should try to explain the evolution of such concept.

On this occasion, as a case study, we will dissect the evolution of the definition of a 'well-ordered set'. To realize how important this concept is to modern mathematics, let us recall, for a moment, that Hilbert included two major issues associated with this question as the first problem in his famous list presented in 1900.[4] Even more significant, Cantor's construction of the concept of a natural number is the reason why must modern textbooks (in almost any branch of mathematics) start by discussing the elementary algebra of sets.[5]

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3. Plato. *The Republic*. Book VII.

4. David Hilbert. "Mathematical Problems." *American Mathematical Society Bulletin* **8** (1902) 437-479. Translated by M. W. Newson. [Originally published as: David Hilbert. "Mathematische Probleme". *Archiv der Mathematik und Physik* **III, 1** (1900) 44-63 and 213-237].

5. Georg Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers*. New York: Dover. 1915. Introducción y traducción de Philip Jourdain. §5. Pages 97-103 .

A modern textbook introduces or defines this notion by saying ‘that any ordered set is a well ordered set if any subset, different from the empty set, has a first element.’[6] Some examples of well-ordered sets include:

1) The set of all natural numbers in their natural order of precedence

$$\{1, 2, 3, ..., n, ...\};$$

2) the union set of sets of powers of natural numbers

$$\{1, 2, 3, ..., n, ...; 1, 4, 9, ..., n^2, ...; 1, 8, 27, ..., n^3, ...; ...; 1^n, 2^n, 3^n, ..., n^n, ...; ...\}.$$

But the set of all real numbers, in its natural order of precedence, does not constitute a well-ordered set, because one cannot name the immediate successor of a specific number and, therefore, there would be subsets without a first element. The definition of a well-ordered set does not include any sophisticated new elements so; it is to possible that, once the professor has provided some examples, the concept will become clear to the student. In fact, some students, as a natural reaction, once understood the idea, will try to memorize the wording of the definition. But, then, if we are acquainted with some of the traditional explanations of memorized knowledge, we know that, eventually, students will forget the precise original wording.

In the case of the definition of a well-ordered set, one may track the original wording to the works of Georg Cantor (1845-1918), in particular, to a paper published in 1883 and briefly known by specialists as ‘the Beiträge’.[7] The wording of the original definition contained three conditions and, in those days,

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7. Georg Cantor. “Foundations of a General Theory of Mannifolds.” *The Campaigner (Journal of the National Caucus of Labor Committees)* **9**$_{1\text{-}2}$ (1976) 69-96. [Originally published as: Georg Cantor. “Grundlagen einer Allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen”. Leipzig: B. Teubner. 1883. Also published as: Georg Cantor. “Ueber unendliche, lineare Punktmannigfaltigkediten”. *Mathematische Annalen* **21** (1883) 545-583; reprinted in Georg Cantor. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*. Editor Ernest Zermelo. 1932. Berlin: J. Springer. Pages 165-208].

even professional mathematicians had difficulties understanding it. We know, at least, two individuals (Hadamard and Burali-Forti) who confused the original definition in professional publications.[8]

So, if at the turn of the XXth century, professional mathematicians had difficulties understanding the wording and the meaning o this new concept, it is natural to imagine eventual predicaments for students. It is not surprising, then, that some mathematicians searched for simpler definitions. For example, in 1904, Philip Jourdain (1879-1919) introduced an equivalent definition by saying that "in order that a simply ordered [set] *M* should be well ordered, it is necessary and sufficient that *M* should contain no ([subset]) of type $*\omega$".[9] But there are, at least, two objections to this new definition. First of all, now, in this case, professors have to explain the students the meaning of ω and of $*\omega$. Second, the definition includes a negative condition. This new definition is a clear example that not all new contributions are necessarily accepted by the community and included in the main stream of a discipline.

It is possible that Jourdain was unaware that, already in 1900, Hilbert had introduced an equivalent definition with a much simpler wording. In his famous lecture of 1900, Hilbert had presented the concept of a well-ordered set in very similar terms to those we find in a modern textbook.[10] By now, the students would have acquired an understanding of how the wording of such definition evolved

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8. Jules Hadamard. "Sur certain applications possibles de la théorie des ensembles." *Verhandlungen des Ersten Internationalen Mathematiker Kongresses in Zurich vom 9. bis 11 August 1987* [Leipzig. 1898]; and, Cesare Burali Forti. "Una questioni sui numeri transfiniti". *Rendiconti del Circolo Matematico di Palermo* **11** (1897) 154-164

9. Philip Jourdain. "On the transfinite cardinal numbers of well ordered aggregates." *Philosophical Magazine* **VI, 7** (1904) 65-66].

10. David Hilbert. *Op. Cit.* Page 446.

and, even more, who was the first to introduce such a concept and where and when he did it. Perhaps, one could have even included very colorful personal anecdotes of all those involved in the evolution of this particular concept. For example, it is well known that some items of the popular literature described Cantor as mentally instable.[11] The concerned teacher might also include some of the well-known anecdotes concerning Hilbert. It is also possible to include Jourdain's personal gossip and those surrounding him.

But, most important and pedagogical useful, during the lecturing process, we should critically analyzed *why* this definition was required. Through this process, we will show our students that this is not a simple case of esthetic appeal or creative, but unfounded, geniuses. In order to clarify this concept, we might discuss, supported by the usage of primary sources, *how* Cantor had visualized the development of the theory of transfinite (those that are 'beyond' the finite) numbers and *what* was the role-played by this concept.

Cantor did not conceive these numbers as isolated pathological cases. It is extremely important for our students to realize that once Cantor found some of the first examples and properties of these transfinite numbers, he visualized a complete arithmetical theory for them, including the operations of addition, multiplication and exponentiation. Of course, these new transfinite numbers had different properties from the finite ones and, therefore, had different arithmetical properties. Cantor *needed* to order these numbers in a simple sequence to show that these transfinite numbers formed a solid and consistent system, in the same

11. See, among others: Eric T. Bell. *Men of mathematics.* New York: Simon & Shuster. 1937. Pages 555-579; Morris Kline. *Mathematical thought from ancient to modern times.* New York: Oxford. 1972. Page 995; and, Stuart Hollingdale. *Makers of mathematics.* England. Penguin books. 1989. Page 359.

way as the finite cardinal numbers. At this particular point, the tricotomy law [if α and β are any two transfinite numbers then one and only one of the three following relations holds among them: $\alpha = \beta$, $\alpha \prec \beta$ or $\alpha \succ \beta$] played a fundamental role. Once Cantor had established a well-ordered sequence of transfinite numbers then, and only then, he was able to develop this arithmetic. Of course the theory (and its history) is much more sophisticated and complex. As we know, Cantor subsequently proved that, among other questions, that

$$\aleph_o + \aleph_o = \aleph_o,$$

$$\aleph_o \cdot \aleph_o = \aleph_o,$$

$$2^n \succ n.$$

On the other hand, students, of course, should realize that Cantor was unable to develop and proof all the characteristics and properties he had visualized. For example, Cantor was unable to proof that every ordered set was also a well-ordered set. In the same way that Galileo was unable to explain the tides and Newton was unable to explain how light travels.

To conclude, by looking at the history of mathematics, we may show our students *how* some scholars advanced notions that they *need* for their research. This may also provide the student, at the beginning of a course, with a panoramic view of *where* he is going and *why* is he taking such a path.