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The set-theoretic paradoxes

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1 CANTOR'S PARADOXES

Sometime between 1883 and 1899, Georg Cantor came across what we now call the paradoxes of the great cardinal and ordinal numbers. How, when and why did he discover them? Cantor himself claims that he had *intuitively* understood them since 1883. In spite of his own assertions, it is difficult to establish conclusively from the surviving sources the events that caused him to discover the paradoxes, his reasoning, or even the date of the discovery. First, based on cogent grounds, some historians have speculated that Cantor *formally* encountered the paradoxes while working on one of the following questions: (a) an attempt to prove that every cardinal number was an aleph; (b) an investigation of the comparability of the powers of all transfinite sets; or, perhaps, (c) an enquiry into the more general question of the well-ordering of all sets. Second, others have thought that the discovery of the paradoxes arose from Cantor's 1891 theorem asserting that the cardinality of a power set is always greater than the cardinality of the set itself. A third possible explanation rests on his composition of a clear and final presentation of his arithmetic of transfinite cardinal and ordinal numbers.

Historians have suggested that Cantor discovered the paradoxes in one of the following years: 1883, 1890, 1892, 1895, 1896 and 1899. There is evidence both for and against all of these dates. The clues contained in Cantor's public works and private correspondence are brief and scarce. From his correspondence with Richard Dedekind (Grattan-Guinness *1974*: 126–8; van Heijenoort *1967*: 113–17), we learn that Cantor concluded that it was necessary to distinguish between two kinds of multiplicity: 'absolutely infinite' (or 'inconsistent') multiplicities and 'consistent' multiplicities (or 'sets'). He realized this limitation of size was mandatory when he became aware that, for some multiplicities, 'the assumption that *all* of its elements "are together" leads to a contradiction' (van Heijenoort *1967*: 114). Examples of inconsistent multiplicities were the totality of everything thinkable and the system of all cardinalities. Cantor used an argument similar to the paradox now associated with the concept of the greatest of all ordinal numbers in his attempts to show that Ω (the system of all ordinal numbers) was an inconsistent multiplicity. Cantor's argument ran as follows:

> If Ω′ [Ω′ results from adding 0 to Ω] were consistent, then, since it is a well-ordered set, there would correspond to it a number δ greater than all numbers of the system Ω; but the number δ also occurs in the system Ω, because this system contains *all* numbers; δ would be greater than δ, which is a contradiction. (van Heijenoort *1967*: 115)

Contrary to popular belief, the discovery of the paradoxes did not result in the logical refutation of Cantorian set theory, nor was it imperative to re-examine the theory's foundations. Cantor, for example, regarded the paradoxes as a positive contribution to his studies, from the mathematical, philosophical and theological points of view. In fact, his theological convictions allowed him to metamorphose the (theological) Absolute into his mathematical absolute of collections too large to be sets (Dauben *1979*: 294–7).

Almost immediately afterwards, perhaps coincidentally, other philosophers and mathematicians came across similar paradoxes – that is, arguments that derive self-contradictory conclusions by valid deductions from apparently acceptable premisses. In particular, in an unpublished letter of 1898 to Cantor, E. H. Moore found the paradox expressible in terms of the greatest of all ordinal numbers. Edmund Husserl discovered the paradox of the greatest cardinal number, and Ernst Zermelo came across the paradox known today as Russell's contradiction (see Section 2). The investigations leading to these discoveries were not undertaken to criticize or discredit Cantor's thought. Prior to 1903, David Hilbert, Dedekind, Zermelo, E. H. Moore, Husserl and Giuseppe Peano, among others, were acquainted with these paradoxes or others like them. Nevertheless, none of these philosophers and mathematicians published any comment on the paradoxes, and the mathematical community at large ignored them.

2 RUSSELL'S CONTRADICTIONS

When Bertrand Russell arrived at Cambridge University in 1890, at the age of 18, he was already interested in studying the foundations of mathematics. There he studied mathematics (the first three years) and philosophy (a fourth year). Still under the influence of some of his philosophy professors, Russell made his first unfulfilled attempt at writing a book on the principles of arithmetic in a manuscript of 1898. There were two more unsuccessful endeavours before his meeting with Peano at the First International Congress of Philosophy at Paris in July 1900.

In October 1900, Russell sat down to begin writing what would eventually turn out to be the book *The Principles of Mathematics* (*1903*). But even before he had finished the manuscript, Russell thought that he had found a 'fallacy' in Cantor's theory of transfinite numbers. In Russell's words:

> There is a greatest of all infinite numbers, which is the number of things altogether, for every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greatest number, ... But in this one point, the master has been guilty of a very subtle fallacy . . . . (Russell *1901*: 95)

Over time, slowly and unconsciously, Russell metamorphosed Cantor's apparent mistake into inescapable difficulties (Coffa *1979*, Grattan-Guinness *1978*).

There are sources, both unpublished and published, showing that around May and June 1901 Russell was involved intellectually with not one but three potential 'contradictions' (as he often called them). Between May 1901 and May 1903, while writing the final version of *The Principles of Mathematics*, he would come to think of two of these three arguments as irresoluble *contradictions* (i.e. statements containing propositions one of which denies or is logically at variance with the other).

More precisely, between January and June 1901 Russell had shifted his attention from his original difficulty involving Cantor's supposed mistake to an argument concerning the class of all classes which are not members of themselves. For Russell, perhaps because of its technical simplicity and also because he was unaware that he was discovering the other contradiction, the most important of the first two contradictions was the one described in terms of the class of all classes which are not members of themselves. The contradiction arose when he asked whether this class was a member of itself, or not. Each alternative led to its opposite.

The second contradiction also arose from Russell's original concern. Russell's final version reads:

> Briefly, the difficulty may be stated as follows. Cantor has given a proof that there can be no greatest cardinal number, and when this proof is examined, it is found to state that, if u be a class, the number of classes contained in u is greater than the number of terms of u, or (what is equivalent), if α be any number, 2^{α} is greater than α . . . . The difficulty arises whenever we try to deal with the class of all entities absolutely or with any equally numerous class . . . . (Russell *1903*: 362)

As mentioned above, Russell did not realize that he was discovering this second and independent contradiction. The process of revelation was mostly unconscious. Perhaps he thought this second contradiction was too obvious, and Cantor could not have missed it.

A third contradiction eventually emerged from *The Principles of Mathematics*, which has come to be known as the Burali-Forti paradox. However, Russell originally considered it as two different contradictions. According to him, on the one hand, a form of the Burali-Forti paradox involved contradictory theorems by Cantor and Cesare Burali-Forti on the comparability of transfinite ordinal numbers. On the other hand, in the same paragraph (Russell *1903*: 323), he also indicated the existence of yet another contradictory result discovered by Burali-Forti. Nevertheless, he also initially thought that there was a way to deny a premiss of Burali-Forti's original argument. In time, partly due to the works of Henri Poincaré and Philip E. B. Jourdain, among others, Russell's second form of the contradiction materialized as the paradox of the greatest ordinal number, and it is known today essentially in the same form as used by Burali-Forti in his *reductio ad absurdum* proof of the incomparability of transfinite ordinal numbers published in 1897 (Copi *1958*: 281, 284).

3 OTHER PARADOXES

In the mathematical literature it has been suggested that the three paradoxes described above led directly to many others. This claim, however, has recently been disputed; there may not have originally been such a direct and strong connection.

In December 1904, Russell received a letter from G. G. Berry, then a part-time librarian at the Bodleian Library at Oxford, communicating his dissatisfaction with *The Principles of Mathematics*. In the first place, Berry rejected Russell's solution to the Burali-Forti paradox, claiming that it was easy to prove that the series of all ordinal numbers was a well-ordered set (and that Cantor had actually done it). Second, he described a contradiction (which he thought was too obvious to have passed unnoticed) in terms of 'the least ordinal number not definable in a finite number of words'. But this expression defined that specific number in and of itself.

Soon after, in a letter of June 1905 addressed to the editor of the *Revue Générale des Sciences Pures et Appliquées*, the French mathematician Jules Richard described what he labelled a 'contradiction' in the roots of set theory. Richard had read some notes published by Jacques Hadamard. These notes mentioned the possible existence of contradictions between, on the one hand, results obtained by Julius König (presented at the Third International Congress of Mathematicians, 1904) and by Zermelo (published in *Mathematische Annalen* in 1904) and, on the other hand, theorems on ordinal numbers by Cantor and Burali-Forti. Richard's contradiction was also formulated in terms of a number not definable in a finite number of words. Richard's finding has some remarkable features. He explicitly claimed to have discovered a contradiction – not a paradox, nor an antinomy. He claimed to have found the contradiction in the mere roots of the concept of set. Later, he suggested that the source of the contradiction was the notion of infinite set. Finally, and most importantly, Richard provided a construction (using Cantor's own diagonal procedure) of such a number.

At about the same time, other mathematicians discovered additional findings involving numbers not definable in a finite number of words. König attempted to prove that the continuum was not a well-ordered set. In the process of his proof, by *reductio ad absurdum*, he supposed that the continuum was a well-ordered set and arrived at a contradiction. Then, he rejected his original hypothesis and concluded the opposite. Simultaneously, Alfred C. Dixon attempted to prove that it was impossible to well-order a transfinite set by a finite set of rules. He too attempted to carry out his demonstration by a *reductio ad absurdum* argument, and his proof also involved a contradiction concerning a number not definable in a finite number of words. But he too rejected his original hypothesis, and concluded the opposite.

Nevertheless, in 1908 it was Russell, once again, who took König's and Dixon's proofs, detached the submerged contradictions in the arguments by *reductio ad absurdum* and recast the contradictions in their own terms. Russell's formulation runs as follows:

> Among transfinite ordinals some can be defined, while others can not; for the total number of possible definitions is $\aleph_0$, while the number of transfinite ordinals exceeds $\aleph_0$. Hence there must be indefinable ordinals, and among these there must be a least. But this is defined as 'the least indefinable ordinal', which is a contradiction. (Russell *1908*: 223)

Finally, it has recently been disputed whether these paradoxes did lead Zermelo to his axiomatization of set theory (Moore *1978*: 307). On the contrary, unquestionably the paradoxes had a great effect upon the final presentation of Russell and Whitehead's *Principia mathematica* (1910–13), and upon various later researches.

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