Structures in Mathematical Theories

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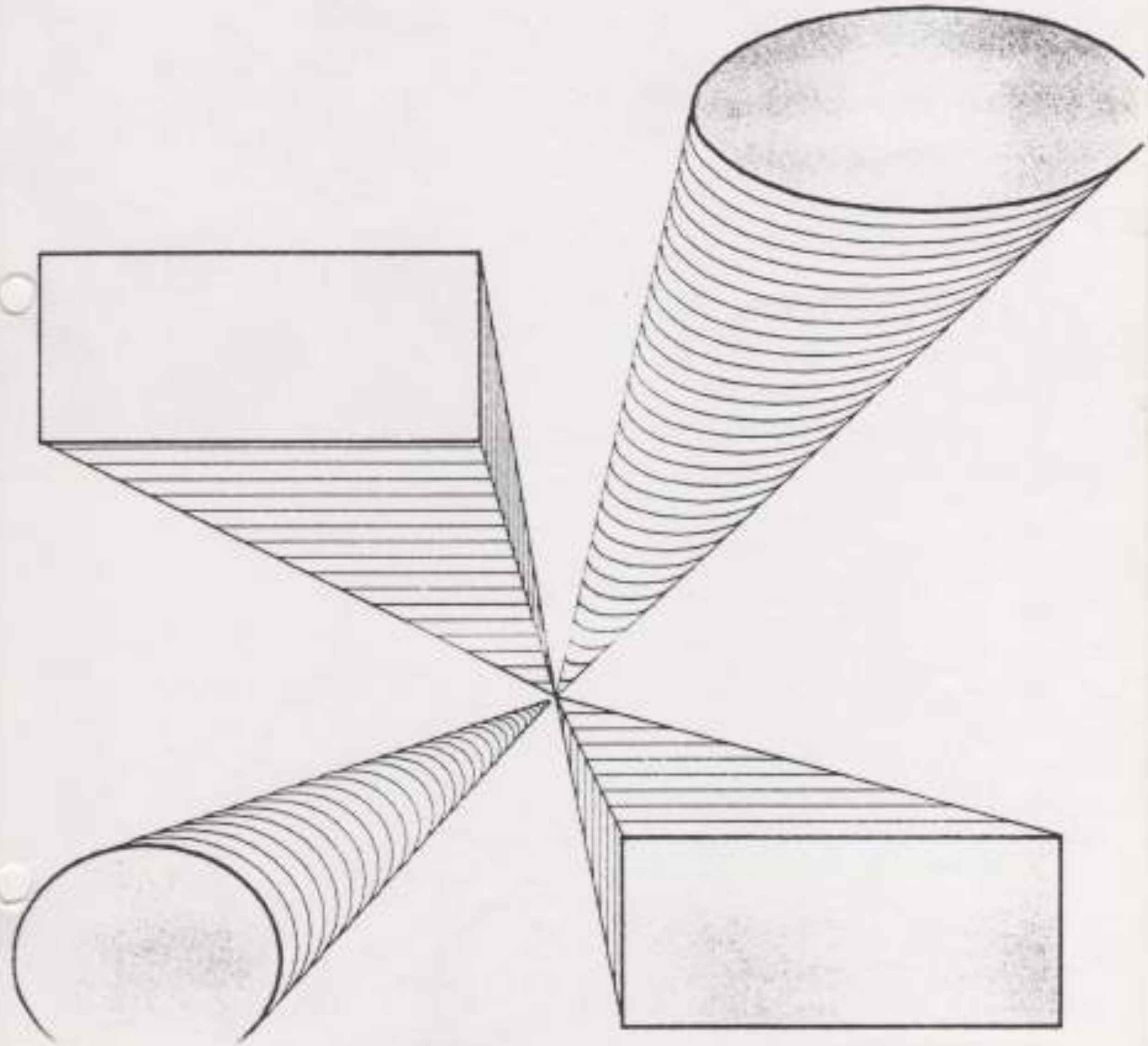

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The Set Theoretic Paradoxes: their influence at the turn of the Century.

INTRODUCTION

A brief and partial summary of a standard interpretation of the origins, development and consequences of the set theoretic paradoxes could be composed using the following basic statements:

1) Most scholars claim that Burali-Forti discovered the inconsistency of the notion of the greatest ordinal number in 1897.

2) Immediately afterwards, dozens of papers appeared dealing with this paradox. And, as a consequence, more such paradoxes were discovered.

3) It has also been said that Cantor came upon similar propositions connected with the greatest cardinal and ordinal numbers in 1899. Moreover, recent studies assert that this discovery came perhaps as early as 1883, or 1895 or 1896 and, therefore, that Cantor anticipated Burali-Forti in his discovery.

4) It has also been equally argued that Russell presented another paradox in *The Principles of Mathematics*. He had discovered this contradiction—as he originally called it—in 1901, and described it in a letter to Frege a year later.

5) Following this line of thought, many historians of mathematics noted the development of a crisis in the foundations of mathematics which resulted from the discovery of the paradoxes. Historians have had no doubt that these paradoxes motivated other mathematicians to develop three philosophical schools in mathematics: *logicism*, *formalism* and *intuitionism*. Struik, for example, asserts [1967, 161] that, "There remained logical difficulties in the theory of transfinite numbers and paradoxes appeared, such as those of Burali-Forti and Russell. This again led to different schools of thought on the foundations of mathematics."

6) Some philosophers have similar ideas about this standard interpretation. Bishop claimed [1975, 509] that the most logical place to begin analyzing the foundations of modern mathematics was with Cantor, "because the paradoxes that arose in Cantorian set theory are what I think really provoked the crisis and the re-examination of the foundations of mathematics that took place. They gave the problem an urgency".

This standard interpretation seems reasonable, as well as apparently chronologically consistent and coherent. Nevertheless, in earlier works, I have criticized the standard interpretation in three ways. First, I argued that paradoxes were not originally encountered as result of the criticism of the theory of transfinite numbers. Second, and perhaps most important, I have shown that it was Russell who first published the so-called Cantor's paradox, his own contradiction of the class of all classes which are not members of themselves, and also gave the first steps necessary for the discovery of the Burali-Forti paradox. All these ideas were contained in Russell's *The Principles of Mathematics*, first published in May 1903. Thirdly, following the dichotomy proposed by Ramsey, I have argued that semantic paradoxes were not a direct product of the logical ones.

As I noted before, it was originally understood that the philosophical schools arouse because of the set theoretic paradoxes. Now, if the new historical reconstruction is plausible, then it is necessary to determine the influence of these 'paradoxes' on the mathematical community after the publication of Russell's *The Principles of Mathematics* in May 1903.

The first publications on Logicism appeared approximately between 1901 and 1902, on Formalism in 1899, and on Intuitionism in 1907. It is clear that members of the first two schools apparently published essays *prior* to the discovery of the 'paradoxes'. This would indicate, at least *prima facie*, that these two schools did not originate as a direct consequence of the discovery of these paradoxes. But the inferences do not follow straightforward. We also know that some of these intellectuals were acquainted with the paradoxes before Russell published his famous book. However, it remains to determine the historical origins of these schools of mathematical thought—in particular, the role the paradoxes actually played. In the discussion that follows, I will sketch some of these basic ideas taking into consideration the new historical account. Instead of closing an area of historical, philosophical and mathematical research, this new reconstruction makes way for an even greater and richer number of new queries. •

LOGICISM

The central thesis of the Russell's work, now known as the *logicist thesis*—and originally developed in his *The Principles of Mathematics*, asserted that mathematics was simply a branch of logic; that is to say, every mathematical concept and theorem could be deduced from more primitive concepts and theorems belonging to formal or symbolic logic.

It is important to note that Russell did not begin his research because of his discovery of 'the Contradiction'. Rather, his interest in the paradoxes was stimulated by his research on the principles of mathematics. Research that he started, at least as early as 1895, with the preparation of his *An Essay on the Foundations of Mathematics*. Furthermore, Russell made three different attempts at a book on the foundations of arithmetic before 1900. The original goal of *The Principles of Mathematics* was not to find a solution for the paradoxes, but to prove that pure mathematics was simply a branch of logic and, along the way to refute Kant's logic, which he had previously admired.

Surprisinly enough, Russell strengthened his logicist thesis while writing the book. Indeed, I contend that Russell had not yet hypothesized a strong logicist thesis when he started writing what turned out to be the final version of this book. When Russell began writing in late October or early November 1900 his fourth attempt at a book on the principles of mathematics, he was planning to discuss the principles of arithmetic in general. In my view, however, he did not realize he could support a stronger thesis until late November 1900, when he read the articles written by Cantor in 1895 and 1897. As I see it, if Cantor had been able to construct the notion of finite cardinal number based upon the concept of set, Russell would have seen that this idea of set might be definable in terms of more primitive notions and concepts. Most allusions to the logicist thesis in the early sections of *The Principles* (Parts III, IV, and V) appear to have been added to the original manuscript. This argues for the view that Russell arrived at the stronger version of the logicist thesis only after he had written some sections of his book. Then, and only then, he added the detailed references to the thesis.

Russell also discovered the 'paradoxes' while writing *The Principles of Mathematics*. However, I argue that Russell developed the logicist thesis to a mature level before Russell discovered the paradoxes. If this were not the case, then a natural question arises: Was it sensible to pursue a stronger philosophical thesis, using logic as a basis, when the roots of logic itself were in doubt?

Even before *The Principles of Mathematics* was published, Russell had already disseminated a germ of his logicist thesis in at least three different articles. Let us not forget that while two thirds of the book were written by the end of 1900, it was not published until May 1903, almost three years later. Two of these essays were printed in Peano's *Rivista di Matematica* (Vol. 7 & 8, 1901 & 1902). The other article was contained in a popular magazine called *The International Monthly* (Vol. 4, 1901). Although this last expository essay provides evidence (in a published document at the time) that Russell was likely to discover the paradoxes, it also included explicit references to his logicist thesis. For example, Russell claims that "the subject of formal logic ... has thus at last shown itself to be identical with mathematics" [Russell 1901, 85]. He also claims that Boole founded pure mathematics, and that in his *Laws of Thought* he treated formal logic which "is the same thing as mathematics" [*Ibid.*, 83]. ▪

FORMALISM

Historians supporting the standard interpretation should find it somewhat more difficult to determine the effect of the paradoxes on the origin of the formalist school. First of all, as Russell pointed out in retrospect, the formalist interpretation was no new at the turn of the century. Secondly, Hilbert, who became known as the leader of this school, published some works explicitly containing some of the premises of formalism as early as 1899 and 1904, even though the basic principles of the program were not systematically developed until the late 1910s and early 1920s.

Some may even object that Hilbert was ever a formalist in the strict sense of the word. Nevertheless, from his 1904 essay in the sphere of number it is possible to assert that for Hilbert mathematics is deprived of concrete content and that it only contains symbolic ideal elements. In this kind of mathematics, terms are essentially symbols and the arguments are formulae including these symbols. The foundations of mathematics is not based on logic, but only in a set of primitive marks and the operations among them.

At the beginning of this discussion, I claimed that the new historical interpretation does not support an analysis of the role of the paradoxes in the origin of these schools prior to 1903, when Russell published them for the first time. However, we also know, according to some historical sources, that Hilbert had knowledge of the existence of the contradictions—as he called them—perhaps, as early as 1896. Is it possible that these ideas on the foundations of mathematics, in particular those on geometry, were provoked by the paradoxes?

If this was the case, then why did Hilbert not turn his attention to the foundations of set theory, which he knew presented great difficulties? It is more reasonable to suppose that he had other reasons for founding geometry on a more solid base. His book was one of several different approaches to develop more profound mathematical rigor, a task that several mathematicians found necessary in the second half of the nineteenth century.

Hilbert's interest in the foundations of mathematics did not arise suddenly in 1896 when he learned of the paradoxes; nor in 1899 with the publication of his book on the foundations of geometry. According to Weyl [1960, 153], a colleague of Hilbert, Blumenthal asserted that Hilbert had claimed as early as 1891, that "it is possible to replace in all geometrical arguments the words point, line and plane for table, chair and mug". Hilbert made this statement when he and Blumental were discussing an article on the role of some theorems by Desargues and Pappus. This statement was apparently made at least four years before Hilbert had knowledge of the paradoxes. Hence one may infer that by 1891, some years before Hilbert had any knowledge either of the set theoretic 'paradoxes' or the resulting crises concerning the foundations of mathematics, he already had in mind some of the elements that would later constitute his formalist program.

A year after the publication of the *Grundlagen der Geometrie*, during the Second International Congress of Mathematicians held at Paris in 1900, Hilbert presented his famous list of problems. The list did not include the set theoretic paradoxes among these twenty three problems. Hilbert may, therefore, have underrated the importance of the 'paradoxes'. That is to say, he believed these arguments did not represent any danger to the consistency of mathematics as a whole. On the other hand, the fact that Hilbert discussed the need of proving the Well-Ordering Theorem along with the Continuum Hypothesis suggested that Hilbert considered set theory to be one of the most important and basic branches of mathematics. Moreover, as it is well known, a great part of the conflict with the Intuitionists occurred because Hilbert refused to be expelled from the paradise Cantor had created for mathematics. The evidence shows that Hilbert considered set theory to be of paramount importance for modern mathematics, but resolving the issues of the paradoxes was not.

It might be argued that even if Hilbert's original interest in the foundations of mathematics did not arise from the discovery of the paradoxes, the work of his disciples did. The most obvious example would be that of Zermelo. According to Kramer [1970, 594], "paradoxes were the motivation for Zermelo, Fraenkel and all the other mathematicians to provide pure set theory with a foundation that would exclude [paradoxes] ... ". This account is generally accepted, in various forms, by those historians and philosophers whose views are consistent with the standard interpretation. In response to this, Moore has convincingly argued [1978, 326] that Zermelo's aim was not to free mathematics from paradoxes, but to show that his "axiomatization was primarily motivated by his proof that every set can be well-ordered, by the controversy which that proof generated, and especially by his Axiom of Choice".

On the other hand, Fraenkel also contributed to the controversy concerning an axiomatization of set theory, apparently under the guidance of Hilbert's principles. An early article of Fraenkel's entitled "The notion "definite" and the independence of the axiom of choice" [1922] contains his criticisms of Zermelo's axiomatization, as well as his proof of the independence of the Axiom of Choice. The article lacks explicit references to the existence of the paradoxes and, apparently, they did not play a relevant role. #

INTUITIONISM

Brouwer defended his doctoral dissertation entitled *Over de Grondslagen der Wiskunde* [*On the foundations of Mathematics*] in late 1906 or early 1907. In this essay he originally expressed some of his ideas concerning the principles of mathematics: a set of ideas that would be perceived later on as the central part of a mathematical theory of knowledge called *Intuitionism*.

For those interested in the origin and metamorphosis of Brouwer's ideas (*ca.*, 1907), there are only a few obscure hints to work with. According to Stigt, one of these clues was the proximity of the publication of Brouwer's *Life, Art and Mysticism* [1905] and his *On the Foundations of Mathematics* [1907]. This dissertation is so intellectually austere that one has to resort to a nonmathematical treatise to comprehend some of Brouwer's mathematical statements. Using the information available in earlier drafts of the dissertation, scholars have "confirmed the very close link between Brouwer's solipsistic and mystical tendencies on the one hand and his philosophy of mathematics and intuitionistic mathematics on the other" [van Stight 1979, 388]. In fact, Stigt has pushed his interpretation even further and asserted that "Brouwer's [illegible] disapproval led to a personal conflict in the mathematician Brouwer, a conflict which he tried to resolve through his own conception of mathematics" [*Ibid.*, 389].

Kant's philosophy plays a fundamental role in Brouwer's intuitionism. According to Brouwer, it is possible to find an earlier formulation of this school in Kant's system and Brouwer believed that he only needed to bring Kant's views up to date. One of the first tasks of the Brouwerian program was to establish the nature of mathematical propositions. Using Brouwer's own terms, "mathematics is created by a free action independent of experience; it develops from a single aprioristic basic intuition, which may be called *invariance in change* as well as *unity in multitude*" [Brouwer 1907, 97]. That is to say, intuitionism posits that mathematics is a basic manifestation of the intellect, made without reference to the 'real world' that surrounds it, a free creation of the human spirit. This 'real world' includes the language used in mathematical discussions, so intuitionistic mathematics has to develop independently of language. Contrary to what Russell, Peano, Cantor, and others had suggested, the natural numbers could not be logically deduced, nor could their existence be postulated, nor could they be constructed upon the concept of set. In Brouwerian terms, the finite cardinal numbers (or natural numbers) are immediately constructed in the mathematician's mind, and their value and certainty are supported by the evidence of intuition.

But let us now return to the question of whether the paradoxes had an important influence in Brouwer's formulation of intuitionism. Firstly, it might be questioned, whether Brouwer knew of the paradoxes when he first developed his ideas. Secondly, according to Brouwer, inconsistencies were based on a mathematically unjustified principle that had to be abandoned. The paradoxes had their roots in the acceptance of actual infinite quantities and, in Brouwer's views, there was no place for these elements in mathematics. Later, Brouwer was to indicate that these contradictory arguments (Russell's contradiction and Burali-Forti's paradox, in particular) do not exist mathematically. Both arguments rested upon linguistic considerations not belonging to mathematics. So paradoxes were only a sign that something was wrong in mathematics. Nevertheless, the difficulties underlying the paradoxes were much more profound and they required a serious reformulation.

I claim that Hilbert's 1904 lecture (at the Third International Congress of Mathematicians) was addressed to Frege, Russell and others who believed that mathematics was a branch or part of symbolic logic. I contend that Brouwer was also critical of Russell's overall program. Nevertheless, Brouwer did not use his entire dissertation to disproving Russell's thesis. Although Brouwer considered his opinions on Russell and the role of logic in mathematics to be of significance for the philosophy of mathematics, he regarded them as belonging to a broader and overly ambitious philosophical program. In fact, Brouwer was willing to sacrifice (and scrap) his criticism of Russell in his dissertation if other sections—that he believed were more important—were not omitted, as his adviser was suggesting. ▌

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