Historia Mathematica 8 (1981) 319-350

BURALI-FORTI'S PARADOX: A REAPPRAISAL OF ITS ORIGINS

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SUMMARIES

Using both published and unpublished letters and manuscripts, this article shows that Burali-Forti's paradox, which has long been regarded as the first of the set-theoretical paradoxes to be discovered, was not created by either Burali-Forti or Cantor. It arose gradually and began to take recognizable form only in Russell's The Principles of Mathematics *of 1903. Russell's long-standing predisposition to seek paradoxes was a vestige of the Kantian and Hegelian philosophical traditions in which he was schooled. Between 1904 and 1906, Burali-Forti's paradox was nurtured by Jourdain and Poincaré, both of whom considered it to be more fundamental than Russell did. To the end, both Burali-Forti and Cantor maintained that there was no such paradox.*

Depuis longtemps, les historiens ont cru que le premier des paradoxes ensemblistes jamais découvert était le paradoxe de Burali-Forti. En s'appuyant sur des lettres et documents publiés ou inédits, cet essai montre que, ni Burali-Forti ni Cantor n'ont créé ce paradoxe. Bien autrement, ce paradoxe émergea petit à petit et ne prit vraiment forme que dans The Principles of Mathematics *de Russell en 1903. La traditionnelle prédilection de Russell à chercher des paradoxes découle des traditions philosophiques kantienne et hégélienne dans lesquelles il fut éduqué. Entre 1904 et 1906, c'est Jourdain et Poincaré qui ont développé le paradoxe de Burali-Forti; tous deux le considéraient comme plus fondamental que Russell ne le croyait. Jusqu'au bout, Burali-Forti et Cantor insistaient qu'il n'y avait pas de tel paradoxe.*

0315-0860/81/030319-32$02.00/0

Usando cartas y manuscritos inéditos y publicados, este artículo muestra que la paradoja de Burali-Forti, la cual ha sido observada por mucho tiempo como la primera de las paradojas de la teoría de conjuntos en haber sido descubierta, no fue creada ni por Burali-Forti ni por Cantor. Esta surgió gradualmente y empezó a tomar forma reconocible en el The Principles of Mathematics *de Russell de 1903. La predisposición de Russell por buscar paradojas fue un vestigio de las tradiciones filosóficas kantianas y hegelianas en las que fue educado. Entre 1904 y 1906, la forma de la paradoja maduró a través del trabajo de Jourdain y Poincaré, quienes la consideraron mucho más fundamental de lo que la había considerado Russell. Burali-Forti y Cantor siempre pensaron que no existía tal paradoja.*

In history we need to construct ignorance situations for our historical figures, that is, branches of relevant knowledge constructed between their period and ours of which they were essentially ignorant.

--I. Grattan-Guinness [1979, 483]

1. INTRODUCTION

Historians and mathematicians have long agreed that the set-theoretic and logical paradoxes, discovered near the end of the 19th century, quickly revolutionized the foundations of mathematics. Furthermore, historians as diverse as Bochenski [1961, 388], Bourbaki [1969, 46], Kline [1972, 1003], W. Kneale and M. Kneale [1962, 652], and van Heijenoort [1967, 104] have all stated that the Italian mathematician Cesare Burali-Forti published the first such paradox in 1897. Thus it may surprise the reader to learn that Burali-Forti did *not* publish a paradox in 1897, or at any other time [1].

Where and how, we may then ask, did Burali-Forti's paradox originate? While it has been suggested that Cantor first formulated Burali-Forti's paradox, this answer, too, requires modification (see Sections 4 and 5). To answer our question, we have found it necessary to examine the historical relationship between Burali-Forti's paradox and the other paradoxes, such as Russell's, which arose during the same period. Our research has led us to conclude that Burali-Forti's paradox originated not all at once, but little by little, primarily through the labors of Bertrand Russell. Moreover, the emergence of Burali-Forti's paradox depended on a "Gestalt switch," a reversal of figure and ground, the essential elements of which had been present for some time [4]. For these reasons and others, we have abandoned the traditional emphasis on who originally stated the paradox in question. In fact, both what was stated and what was meant by the paradox underwent a transformation in the hands of English, French, German, and Italian mathematicians during the decade following 1897. The process by which Burali-Forti's paradox originated, diffused, and metamorphosed into a form recognizable to mathematicians today was essentially complete by 1907.

In the present article, we distinguish sharply between the terms *paradox* and *contradiction*, a distinction not necessarily made by mathematicians of the period. In his argument Burali-Forti arrived at both a conclusion and its negation, hence at a contradiction. By contrast, a paradox (as we shall use the term) is an argument which ends in a contradiction although all of its premises and modes of reasoning are prima facie acceptable. In particular, a paradox requires that the one who discovers it give up a premise or mode of reasoning that he has previously accepted as correct. Thus a paradox appears "paradoxical" to its author if he is uncertain which of his premises to abandon. To Burali-Forti, on the other hand, it was evident that the trichotomy of ordinals was unacceptable as a premise and was, in fact, false. Therefore Burali-Forti's argument did not constitute a paradox when he wrote his article [1897a], and he did not find his result threatening in any way. Yet through the labors of various mathematicians, his argument would eventually be transformed into a paradox.

2. BURALI-FORTI

From the beginning, two mathematicians inspired Burali-Forti to investigate the foundations of mathematics: Giuseppe Peano and Georg Cantor. During the academic year 1893-1894 Burali-Forti lectured at the University of Turin on Peano's mathematical logic and soon published these lectures as a slim volume [1894a]. At the same time he completed an article [1894b] expressing Cantor's theory of transfinite ordinals in Peanoesque symbolism.

While the article faithfully mirrored Cantor's theory in most respects, Burali-Forti misconstrued the definition of *well-ordered set*. A decade earlier Cantor had first introduced this notion in the following fashion: *An ordered set is well-ordered if it contains a first element, if any element with a successor has an immediate successor, and if any finite or infinite set of elements which has a successor also has an immediate successor* [1883a, 548-549]. Perhaps because of the meandering fashion in which Cantor had stated this definition, it was misunderstood by Jacques Hadamard [1897, 301] and Burali-Forti, among others. In particular, Burali-Forti retained only the first two clauses of Cantor's definition: *An ordered class is well-ordered if*

> *(a) it contains a first element, and*
> *(b) every element with a successor has an immediate successor.* [1894a, 172] [3]

Thus Burali-Forti did not require, as Cantor had done, that every infinite set of elements with an upper bound have a least upper bound. This misunderstanding is vital if one wishes to grasp why Burali-Forti did not discover Burali-Forti's paradox.

After reading Cantor's seminal article [1895] on the theory of cardinal numbers and order-types, Burali-Forti investigated two problems in cardinal arithmetic that Cantor had left open: to prove the Equivalence Theorem (if $\bar{\bar{A}} \le \bar{\bar{B}}$ and $\bar{\bar{B}} \le \bar{\bar{A}}$, then $\bar{\bar{A}} = \bar{\bar{B}}$) and to establish the trichotomy law for cardinals ($\bar{\bar{A}} < \bar{\bar{B}}$ or $\bar{\bar{A}} = \bar{\bar{B}}$ or $\bar{\bar{B}} < \bar{\bar{A}}$). Although Burali-Forti made little progress on the first problem (resolved shortly thereafter by Felix Bernstein), he deduced the trichotomy of cardinals from two new postulates which he introduced:

> *(1) If A is a class of non-empty classes, then the cardinal number of A is less than or equal to the cardinal number of the union of A.*
> *(2) For any classes A and B, there is a function $f: A \to B$ which is either one-one or onto.* [1896, 236-237] [4]

Next, Burali-Forti wished to determine whether trichotomy also held between all pairs of ordinal numbers. In fact, his famous article of 1897 was an attempt to refute trichotomy in this case by exhibiting two ordinal numbers α and β such that neither $\alpha < \beta$ nor $\alpha = \beta$ nor $\beta < \alpha$. More precisely, he intended to define two order-types, α and β, of "perfectly ordered classes" which violated trichotomy. Here his misunderstanding led him astray, for he continued to believe that an ordered class A was well-ordered if and only if it satisfied conditions (a) and (b) stated above. As a result he introduced the more restrictive concept of *perfectly ordered class*, which satisfied not only (a) and (b) but also a new condition:

(c) for every x in A, if x has an immediate predecessor then there is some predecessor y of x such that y has no immediate predecessor and that only finitely many elements of A are between y and x. [1897a, 157]

When Burali-Forti completed his article in February 1897, he thought that every perfectly ordered class was well-ordered, but that the converse failed to hold. Thus he believed that the concepts of well-ordered class and of perfectly ordered class were quite distinct. Nevertheless, the relationship between these two sorts of classes was to prove quite problematical.

Burali-Forti began his construction, intended to violate trichotomy, by introducing the class *NO* of all order-types of perfectly ordered classes. He showed that if trichotomy held, then *NO* was itself a perfectly ordered class. Consequently its order-type Ω was a member of *NO*, as was $\Omega + 1$. But if $\alpha \in NO$, then $\alpha \leq \Omega$; hence both $\Omega < \Omega + 1$ and $\Omega + 1 \leq \Omega$ held, a contradiction. He concluded that it was impossible to order the class of all order-types, or even its subclass *NO*--an essential defect which, he argued, kept *NO* from serving as a "standard class" for the order-types in the way that the natural numbers did for finite classes [1897a, 163-164]. Nevertheless, he did not think that he had found a paradox in Cantor's work, but rather that he had shown the theory of order-types to contain an essential limitation. Although now we can easily rephrase Burali-Forti's argument in terms of well-ordered sets, and so give rise to a paradox, Burali-Forti never did so. The only contradiction in his paper was the usual one found in any proof by *reductio ad absurdum*.

Later in 1897, Burali-Forti read Cantor's article [1897, 216] which established that the order-types of well-ordered sets satisfy trichotomy. What Burali-Forti concluded was, not that the theory of transfinite ordinals generated a paradox, but that he had misread Cantor's definition of well-ordered set. In October Burali-Forti composed a brief note [1897b] which rectified the error and stated Cantor's full definition of well-ordered set, discussed above. There he emphasized as well that he had not made use of well-ordered classes in his earlier paper [1897a], nor of classes satisfying conditions (a) and (b), but solely of perfectly ordered classes. Having now defined well-ordered classes correctly, he concluded that (contrary to his earlier claim) every well-ordered class is perfectly ordered, but that some perfectly ordered classes are not well-ordered. Trichotomy held for the order-types of well-ordered classes but not for those of perfectly ordered classes. No disagreement, he believed, arose between his results and those of Cantor.

This interpretation of Burali-Forti's work was adopted as well by his compatriot Giulio Vivanti, who wrote the abstract for Burali-Forti's articles of 1897 which appeared in the

Jahrbuch über die Fortschritte der Mathematik in 1900 [Vivanti 1900, 62-63]. No paradox was mentioned in that abstract, and no other reference to Burali-Forti's articles [1897a,b] seems to have been published prior to 1902. Thus, contrary to the interpretation usually accepted today, it is false that Burali-Forti's paper "immediately aroused the interest of the mathematical world ..." [Van Heijenoort 1967, 105]. For five years no public reaction occurred at all.

2. RUSSELL AND COUTURAT

What ended the silence surrounding Burali-Forti's articles of 1897? Who recognized after half a decade that Burali-Forti's argument could be transferred to the well ordered sets? Who, indeed, transformed Burali-Forti's argument into a paradox?

In brief, it was primarily Bertrand Russell who brought about this transformation during the period 1900-1903. The first explicit statement of Burali-Forti's paradox, as a paradox, occurred in Russell's book *The Principles of Mathematics* [1903, 323]. By January 1901 Russell had already come close to glimpsing Burali-Forti's paradox of the largest ordinal, the paradox of the largest cardinal, and his own paradox (see [Coffa 1979]). In order to understand what took place, it will be helpful to examine his mathematical activities around this period.

Russell began to search for paradoxes in mathematics much earlier than is usually recognized. His predisposition to invent such paradoxes had its roots in the philosophical antinomies of Kant and Hegel, both of whom deeply influenced his early development as a philosopher. In particular, the introduction to his unpublished Hegelian essay, "On Some Difficulties of Continuous Quantity," shows that this predilection was already present in 1896:

> *But it seemed worthwhile to collect and define ... some contradictions in the relation of continuous quantity to number, and also to show, what mathematicians are in danger of forgetting, that philosophical antinomies, in this sphere, find their counterpart in mathematical fallacies. These fallacies seem ... to pervade the Calculus, and even the more elaborate machinery of Cantor's collections* (Hangon).

Russell had first learned of Cantor's work, and had first become mistrustful of Cantorian set theory, by reading a book on atomism by the French Kantian philosopher Arthur Hannequin [1895, 48-71].

There Cantor's researches on the continuum of real numbers were analyzed at length and condemned as contradictory. In his review of this book Russell regarded the contradiction, which Hannequin had detected in Cantor, as merely Kant's second antinomy from the *Critique of Pure Reason*: every composite substance both is, and is not, made up of simple parts [Russell 1896, 412].

Within the next three years Russell had found in Cantorian set theory a paradox close to that of the largest cardinal. For during 1899 he undertook a draft of *The Principles*, and in the outline for Part II he wrote:

> Chapter VII. Antinomy of Infinite Number. *This arises most simply from applying the idea of a totality to numbers. There is, and is not, a number of numbers. This* [and] *causality are the only antinomies known to me. This one is more all-pervading.... No existing metaphysic avoids this antinomy.*

It is likely that Russell developed the "antinomy of infinite number" by reading Leibniz. At this time Russell was preparing his book, *A Critical Exposition of the Philosophy of Leibniz*, for publication, and in an appendix of excerpts from Leibniz' writings on logic he quoted that "the number of all numbers implies a contradiction" [Russell 1900, 244]. Here was a seed that could grow into the paradox of the largest cardinal. Yet in 1899 Russell lacked a method for making this claim rigorous and so obtaining a genuine paradox. Such a method appeared the following year when he made the acquaintance of Peano's symbolic logic.

During July 1900 Russell attended the first International Congress of Philosophy in Paris and was greatly impressed by Peano. After studying his writings in August, Russell began to extend the Peanesque symbolism to his own logic of relations. In October, Russell wrote another draft of *The Principles*. This draft, containing most of the final version of 1902, was completed during December 1900 [1967, 145].

At that time Russell discovered what he believed to be a number of errors in Cantor's writings. On 8 December 1900 he corresponded with his friend, the French philosopher Louis Couturat, about these supposed errors. One of them concerned Cantor's theory of real numbers, but more important to us here was what Russell wrote about an "error" involving cardinal numbers:

> *I have discovered a mistake in Cantor, who maintains that there is no largest cardinal number. But the number of classes is the largest number. The best*

of Cantor's proofs to the contrary can be found in Jahresb. d. deutschen Math. Ver'g., I, 1892, pp. 75-78 [Cantor 1891]. In effect it amounts to showing that, if u is a class whose number is α, the number of classes included in u (which is 2^u) is larger than α. The proof presupposes that there are classes included in u which are not individuals [members] of u; but if u=Class, that is false: [for] every class of classes is a class. [5]

Why Russell believed Cantor's proof to make such a presupposition is not at all clear.

This passage contains all the ingredients that Russell needed to create the paradox of the largest cardinal. Intriguingly, he did not believe at the time that a paradox was present or even that Cantor's theory was endangered in some way. Instead he argued that Cantor's theorem [1891]--for every set *A*, the set of all subsets of *A* has a higher power than *A*--did not hold when *A* was the class of all classes, and hence that Cantor had erred in denying the existence of a largest cardinal [6]. Thus it was not a new discovery, but a shift in how he perceived an argument which he already possessed, that later led Russell to formulate clearly and convincingly the paradox of the largest cardinal.

Replying to Russell's letter on 3 January 1901, Couturat introduced a new subject into the discussion: Burali-Forti.

The errors in Cantor which you pointed out to me seem very interesting; but I am far from this subject, and, immersed in Leibniz, I do not have the leisure to study it. The infinite lends itself so easily to paralogisms! Burali-Forti has claimed to prove that it is false to assert for order-types that:

$$(a = b) \cup (a < b) \cup (a > b)$$

and consequently that they do not form a well-ordered class. His reasoning is more specious than convincing. ("Una questione sui numeri transfiniti," in Rendiconti del Circolo matematico di Palermo *28 March 1897 (Vol. XI). I can lend you the article if you wish.) I wonder whether one can consider the class of all possible classes without some sort of contradiction.* [Appendix 1]

Thus Couturat remained dubious that Burali-Forti had shown the trichotomy of ordinals to be false, but he was equally dubious of Russell's class of all classes.

Although he had not known of Burali-Forti's article, Russell had reached a similar conclusion. In his letter to Couturat of 17 January, Russell wrote:

> *I agree with Burali-Forti that it is false to assert*
>
> $$a = b . \cup . a < b . \cup . a > b$$
>
> *for order-types. What is more, I suspect that one cannot assert the same thing for cardinal numbers either. Cantor's arguments on this subject are not conclusive. I would be very grateful if you would lend me Burali-Forti's article, which I have not seen.* [Appendix 2]

When Couturat and Russell used the term "order-type" here, it is probable that they meant "ordinal." For the familiar order-types ω and $*\omega$ (of the positive integers with <, and >, respectively) certainly violated trichotomy if order-types in general were intended. Unfortunately, Russell's letter stated no reason why he denied the trichotomy of ordinals. It is significant that he also doubted the trichotomy of cardinals, which Burali-Forti had deduced from new axioms in a paper [1896] unknown to Russell at the time. For the trichotomy of cardinals was intimately related to the question of well-ordering an arbitrary set, a subject to which we shall turn in the next section.

On the other hand, Russell vigorously defended his class of all classes:

> *If you grant that there is a contradiction in this concept, then the infinite always remains contradictory, and your work as well as that of Cantor has not solved the philosophical problem. For there is a concept* Class *and there are classes. Hence* Class *is a class. But one can prove (and this is essential to Cantor's theory) that every class has a cardinal number.* [Appendix 2]

Russell noted further that "no contradiction results" from *Class* (the class of all classes) since, as he had remarked in his previous letter, Cantor's proof of 1891 did not apply to it. Yet Russell remained vague as to why Cantor's proof did not apply in this case as well (and thus engender a paradox).

On 27 January 1901 Couturat had lingering doubts about *Class*: "Is the class *Class* determined, closed so to speak, in such a way as to possess a cardinal number?" [Appendix 3]. Russell responded on 1 February that *Class* was indeed well defined since for any *x* it was determined whether or not *x* belonged to *Class*. He added that Burali-Forti's papers had arrived, which he read "with great interest."

Thus by early in 1901 Russell had come close to creating both the paradox of the largest cardinal and that of the largest ordinal. Although neither paradox crystallized at the time, he continued to ponder the seeming contradictions of the infinite. In a popular article he discussed Zeno's paradoxes and also commented on Cantorian set theory:

> *On the subject of infinity it is impossible to avoid conclusions which at first sight appear paradoxical, and this is the reason why so many philosophers have supposed that there were inherent contradictions in the infinite. But a little practice enables one to grasp the true principles of Cantor's doctrine, and to acquire new and better instincts as to the true and the false.* [Russell 1901, 101]

Here Russell, while still concerned with "errors" in Cantor's work, revealed again his tendency to seek paradoxes in set theory. Yet, as he later informed Philip Jourdain [1911, 146], he did not discover his own paradox until June 1901; when he composed this article in January, he knew only that Cantor's proof of 1891, that there is no largest cardinal, contained a mistake. On 21 July 1901 Couturat wrote to Russell concerning the article: "It is quite profound beneath its popular guise, and pleased me a great deal. Beneath the paradoxes hide some striking truths" [Appendix 4]. This premonition turned out to be quite accurate, although Couturat did not know that Russell's paradox-hunting had just succeeded.

When Russell, in June 1901, found the paradox which now bears his name, he failed at first to recognize its importance. Apparently he wrote to no one about it at the time. In particular, no reference to it occurs either in his voluminous correspondence with his wife Alys, or in that with Couturat, before mid-1902 [7].

Why did Russell remain silent about his paradox for an entire year? Two letters, one to Alys and one to Couturat, help to explain Russell's silence during the period between discovering his paradox and describing it, first to Peano, and then on 16 June 1902 to Frege [Frege 1976, 211-212]. On 25 June, Russell wrote to his wife: "I have heard from Frege, a most candid letter; he says that my conundrum makes not only his Arithmetic, but all possible Arithmetics, totter" [8]. It was the fact that Frege, whom Russell admired intensely, regarded Russell's paradox as devastating that helped to convince him of its importance. Over the next two months they exchanged nine lengthy letters concerning the paradox and its possible solutions [Frege 1976, 211-227]. Russell then revised the chapter on his contradiction in *The Principles*, a chapter which he had first written not long before [4].

The second letter to clarify Russell's year-long silence contained the first mention of his paradox to Couturat. "I am very busy with my book," Russell wrote of *The Principles* on 29 September 1902,

> *which is being printed slowly, and by vol. II, which I am preparing in collaboration with Whitehead. I do not know what to do about a class of contradictions of which the simplest is this:*
>
> $$w = \hat{x}(x \sim \varepsilon x) \supset : w \varepsilon w . = . w \sim\varepsilon w.$$
>
> *I have tried many solutions without success. One obtains contradictions of this sort by taking Cantor's proof that there is no largest cardinal and applying it to the class of all individuals, or of all propositions, or of all relations. When my book began to be printed, I believed I could avoid these contradictions, but now I see that I was mistaken, a fact which greatly diminishes the value of my book.* [Appendix 5]

This passage suggests strongly that, little by little, Russell came to place his paradox at the center of his foundational concerns. So long as he believed that his paradox could be solved without great difficulty (by deciding which assumption to abandon), there was no reason to treat it as fundamental. Only after failing at numerous attempts to solve it and only after Frege underlined its importance did Russell regard the paradox as crucial.

By focusing on the importance of his own paradox, Russell also shifted his perspective on Burali-Forti's work. Writing in 1902 for Peano's journal *Rivista di Matematica*, he stated a reservation about Burali-Forti's article [1897a]. This reservation constituted Russell's attempt to make sense of both Burali-Forti's and Cantor's articles. For the first time in print, he cast doubt on Cantor's fundamental proposition that every set can be well-ordered [Russell 1902, 33]. On the other hand he had become convinced that Cantor's argument of 1897 establishing the trichotomy of ordinals was correct. Faced with Burali-Forti's article, he decided to deny that the less-than relation on the ordinals was a well ordering, even though he granted that every segment (i.e., the class of those ordinals less than a given ordinal) was well-ordered [1902, 43]. All the same, Russell made no mention of any paradox.

When *The Principles* appeared in 1903, Russell took a similar stand on this matter in all but one respect: He now believed that Burali-Forti's argument generated a paradox. In this way Burali-Forti's paradox of the largest ordinal was finally born:

> *There is a difficulty as regards the type of the whole series of ordinal numbers. It is easy to prove that every segment of this series is well-ordered, and it is natural to suppose that the whole series is also well-ordered. If so, its type would have to be the greatest of all ordinal numbers, for the ordinals less than a given ordinal form, in order of magnitude, a series whose type is the given ordinal. But there cannot be a greatest ordinal number, because every ordinal is increased by the addition of 1. From this contradiction, M. Burali-Forti, who discovered it, infers that of two different ordinals, as of two different cardinals, it is not necessary that one should be greater and the other less. In this, however, he consciously contradicts a theorem of Cantor's which affirms the opposite. I have examined this theorem with all possible care, and have failed to find any flaw in the proof. But there is another premiss in M. Burali-Forti's argument, which appears to me more capable of denial, and that is, that the series of all ordinal numbers is well-ordered. This does not follow from the fact that all its segments are well-ordered, and must, I think, be rejected, since, so far as I know, it is incapable of proof. In this way, it would seem, the contradiction in question can be avoided.* [Russell 1903, 323]

It is important to note that Russell was employing the term "contradiction" in the sense of a paradox and not, like Burali-Forti, as part of an argument *reductio ad absurdum*. Some pages earlier, Russell had repeatedly referred to his own paradox by the term "contradiction" [1903, 101-107]. Through juxtaposing Burali-Forti's and Cantor's articles of 1897, Russell created the paradox of the largest ordinal. Whether he believed that Burali-Forti's perfectly ordered classes and Cantor's well-ordered ones coincide (they do not) or whether he merely transferred Burali-Forti's argument to the well-ordered classes, it was here that Burali-Forti's paradox first reached print. Russell considered himself to have solved it by denying that the class of all ordinals, taken in their usual order, was well-ordered. Nevertheless, influenced by Jourdain, he soon reversed himself and came to regard Burali-Forti's paradox as a significant unresolved problem.

Since Russell composed *The Principles* over a period of several years, he might have conceived of Burali-Forti's paradox--as a paradox--even before he clearly formulated his own. Thanks to material in the Russell Archives, we can be reasonably certain that this was not the case. He completed the final draft of *The Principles* on 23 May 1902 [Russell 1967, 151]. This draft, as well as earlier ones kept in the Russell Archives, contains no mention of Burali-Forti's paradox. In particular the final draft lacks sections 299 and 300 of the printed version, as well as section 301 (quoted above) where Burali-Forti's paradox first occurs. Moreover, the pagination of the final draft is complete. Thus Russell evidently recognized, sometime between June 1902 and May 1903, that Burali-Forti's argument could be transformed into a genuine paradox. Yet, as we shall see, it was not Russell but Jourdain who first underlined the significance of Burali-Forti's paradox.

4. CANTOR AND JOURDAIN

In *The Principles* there occur three sorts of paradoxes. By far the most vital to Russell was his own paradox, together with its variants. A second sort, the breeding ground for the first, involved Cantor's proof of 1891 that there is no largest cardinal; indeed, Russell was the first to publish the paradox of the largest cardinal, which he named Cantor's paradox [10]. Last, there was the paradox of the largest ordinal, which he dubbed Burali-Forti's. Russell regarded his own paradox as the most fundamental and the least technical, since it required less set-theoretic machinery than the other two. Consequently, he concentrated his efforts on it. During 1906 he wrote to his compatriot Jourdain: "In 1905 for the first time I worked seriously at Burali-Forti's contradiction. I had never paid much attention to it before, because it was so much more complicated than mine that it seemed likely either to be soluble in some purely technical way, or to be not soluble until mine had been solved" (quoted in [Grattan-Guinness 1977, 80]). Russell's philosophical notion of complexity, which did not rely on the symbolic complexity of the formal expression of a proposition, will not be analyzed here.

In 1903, after reading *The Principles*, Jourdain himself became intrigued with the paradox of the largest ordinal. In fact, this paradox led him to formulate an argument for Cantor's proposition that every set can be well-ordered. Before sending this argument to Cantor, he showed it to G. H. Hardy, who then wrote Russell about it on 14 October:

> *I enclose a letter I received this morning from Jourdain (whom I think you know), ... The other point about the ℵ's [alephs] is also interesting but I don't think Jourdain is right. For if anything seems obvious, it is that the series of ℵ's does exist. I tried myself when writing my Quarterly Journal paper [1903] to get an argument for every α being in the series (but came to the conclusion that I couldn't) out of Burali-Forti's contradiction. My line of argument was to show that adding one more to the whole series meant nothing if (and only if) it was already similar to the aggregate of all entities.... However if you have (as Whitehead says) solved all the difficulties about the greatest cardinal etc., I suppose you have solved Burali-Forti's contradiction too.* [11]

The issue over which Hardy and Jourdain disagreed was whether the set W of all ordinals, or that of all alephs, exists. For Jourdain had utilized W in his argument that every transfinite cardinal is an aleph (i.e., the cardinal of some well-ordered infinite set) and hence that every set can be well-ordered.

On 29 October Jourdain sent a version of this argument to Cantor. In his reply of 4 November, Cantor revealed in turn that he had sent a similar proof to Richard Dedekind four years earlier and to David Hilbert some three years before that [Grattan-Guinness 1971, 115-117]. Although the letter to Hilbert in 1896 is no longer extant, two letters to Dedekind illuminate Cantor's perspective.

The first of these letters, which Cantor wrote on 28 July 1899, concerned the theory of cardinals: "The main question was whether there exist other [infinite] powers of sets besides the alephs; for two years I have been in possession of a proof that there are no others. ..." [Cantor 1932, 443]. He elaborated in a letter of 3 August [12]. The fundamental distinction which he introduced was that between consistent and inconsistent multiplicities [Vielheiten]: "A multiplicity can be created such that the assumption that all its elements form a 'collection' leads to a contradiction, and thus it is impossible to conceive of the multiplicity as a unity or 'a completed thing.' Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*" (ibid.). On the other hand he designated the consistent multiplicities as sets [*Mengen*]. Here the principal distinction between the two types of multiplicity was one of size.

Inconsistent multiplicities interested Cantor primarily because they suggested a method for establishing his fundamental proposition that every set can be well-ordered. While in an earlier article [1883a, 550] he had claimed that this proposition was "a law of thought," by 1895 he had come to believe that it required a demonstration. The first step in his proof, as communicated to Dedekind, was to show that the collection Ω of all ordinals is inconsistent or absolutely infinite. For if Ω were consistent, then it would have an ordinal number δ greater than every member of Ω; but δ was a member of Ω and so δ < δ, an absurdity. To modern eyes, this passage strongly suggests Burali-Forti's paradox. Yet Cantor did not believe set theory to be threatened in any way by his argument. Thus he did not state Burali-Forti's paradox here--although he possessed all the ingredients needed to do so--because he did not regard it as a paradox. In Cantor's eyes, there was simply the fact that some collections were too large to be sets.

Cantor continued his argument by establishing that every transfinite cardinal was an aleph: If some infinite set V had no aleph as its cardinal, then V would have a subcollection V' in one-one correspondence to Ω. Since Ω was inconsistent, so was V' and hence V as well. But by definition only sets (i.e., consistent multiplicities) possessed cardinal numbers, and hence V had no cardinal. Consequently every transfinite cardinal was an aleph, and every set could be well-ordered.

In 1903, unaware of this correspondence of 1899, Jourdain sent Cantor his similar argument, also distinguishing between consistent and inconsistent collections. Jourdain requested permission to publish Cantor's proof found in his letter of 4 November, but Cantor refused [Grattan-Guinness 1971, 113-117]. Yet Cantor encouraged Jourdain to publish his own version. On 2 December 1903 Jourdain finished the article containing this version and submitted it to the *Philosophical Magazine*, where it was printed in January 1904. There, letting β be the ordinal number of the well-ordered collection W of all ordinals, he stated that the cardinal of W would be $\aleph_\beta$. Then he added:

> *But there can be neither a greatest ordinal nor a greatest Aleph; for, given β, the type of the aggregate (1 ... β) is the ordinal number β + 1, and* $\beta + 1 > \beta$, $\aleph_{\beta+1} > \aleph_\beta$. *This contradiction was first published by Burali-Forti, who concluded from it that one must deny both Cantor's fundamental theorem in the theory of ordinal numbers that: if u_1 and u_2 are any two ordinal numbers, then either*
>
> $$u_1 < u_2, \quad \text{or} \quad u_1 = u_2, \quad \text{or} \quad u_1 > u_2,$$
>
> *and the corresponding theorem for Alephs. This conclusion is, in fact, necessary if one admits Burali-Forti's premisses; and since Cantor's demonstration of the above theorem is beyond all possible objection,*

> *Russell [1903] avoided the contradiction by denying the premiss that the series of all ordinal numbers, arranged in order of magnitude, is well-ordered.... But it appears possible to prove that this series of all ordinal numbers is well-ordered....* [Jourdain 1904, 64-65]

Here Burali-Forti's paradox began to acquire a life of its own, though one based on a misinterpretation. It seems that Jourdain had not read Burali-Forti's article, but simply relied on Russell's somewhat misleading account of it discussed above [13]. In fact, Jourdain paraphrased the version of the paradox found in *The Principles*. Furthermore, in the passage above he echoed Cantor's letter of 4 November 1903 by mistakenly claiming that Burali-Forti had denied the trichotomy of cardinals.

More important in this context is the fact that Jourdain rejected Russell's solution to Burali-Forti's paradox, since the solution denied that *W* is well-ordered. In particular, Jourdain's article established that a class is well-ordered if and only if it has no subclass of type $*\omega$. His result implied that the class *W* of all ordinals was well-ordered, and consequently that Russell's solution to the paradox of the largest ordinal had to be abandoned. Jourdain concluded: "There arises, then, an insuperable contradiction if we speak of *W*, or any similar aggregate, as having a cardinal number or ordinal type" [1904, 66]. In contrast to Cantor's rather vague criterion for an inconsistent collection, Jourdain proposed the following formal definition: An aggregate is inconsistent if it has a subaggregate in one-to-one correspondence with *W*. This definition, potentially quite fruitful, did not resolve the paradoxes in a way acceptable to Russell.

In November 1905, if not sooner, Russell rejected Jourdain's proposition that a class is well-ordered if it lacks a subclass of type $*\omega$. On the other hand, by using transfinite induction, he established Jourdain's conclusion that *W* is well-ordered [1906, 35]. In this way he abandoned his earlier solution to Burali-Forti's paradox. Soon thereafter he inclined more and more toward accepting his "no-classes theory" as a way out of the paradoxes, until he modified it to obtain the theory of types. Yet when he granted in 1905 that *W* is well-ordered, Russell reinforced the conviction, beginning to diffuse through the mathematical community, that Burali-Forti's paradox was a fundamental problem which had to be resolved.

5. THE DIFFUSION OF BURALI-FORTI'S PARADOX

As the year 1905 began, the paradox of the largest ordinal had not yet become a matter of concern to the community of mathematicians interested in set theory. Yet by the end of 1905, a major public debate--involving the paradox of the largest ordinal and related issues--was flourishing in England, Germany, and France. It was Jourdain's article [1904] that first turned Burali-Forti's paradox into a significant issue among mathematicians at large.

Shortly before this diffusion began in earnest, Ernst Zermelo [1904] had published a proof that every set can be well-ordered. Although in time his proof would be recognized as definitive, during 1905 a number of eminent mathematicians insisted that his demonstration had serious flaws (see [Moore 1978]). In fact, several of these mathematicians regarded his proof as essentially identical to that of Jourdain, and so concluded mistakenly that Burali-Forti's paradox was entangled with both proofs. Moreover, although the various disputants disagreed as to how much set theory ought to be preserved, most believed that a substantial amount must be abandoned--perhaps even all the uncountable ordinals and cardinals.

In England, the locus of the debate was the London Mathematical Society. Stimulated by Jourdain's article [1904], the analyst Ernest Hobson offered a caustic critique of Cantorian set theory in a paper read to the Society early in February 1905. Hobson dismissed Jourdain's purported proof that every set can be well-ordered, and rejected Jourdain's proposed solution to Burali-Forti's paradox. Nevertheless, Hobson's account of the previous history of the paradox echoed that already given in Jourdain's article. From the same data Hobson and Jourdain arrived at opposite conclusions. Concerning Jourdain's proposal to distinguish between consistent and inconsistent aggregates, and to refuse the latter both a cardinal number and an order-type, Hobson wrote:

> *This amounts to a denial of the universal validity of the fundamental principle that every ordered aggregate has a definite order type; and yet it is by means of this very principle that the existence of the successive ordinal numbers is regarded as having been established. Each successive ordinal number was defined to be the order type of the ordered aggregate of all the preceding ordinal numbers.* [1905, 171]

Because of Burali-Forti's paradox, Hobson argued, the foundations of set theory must be scrutinized with the greatest of care. He proposed radical surgery. While he acknowledged that ω and aleph-zero were legitimate since there existed countable sets of geometric points with order-type ω and cardinal number aleph-zero, he would not grant the same status to ω_1 and aleph-one, which were uncountable. There was no convincing reason, Hobson insisted, to believe that some uncountable set can be well-ordered. In the same vein he denied the existence of the set of all ordinals as well as that of all cardinals. Moreover, he dismissed both Jourdain's and Zermelo's proofs, which he found quite similar, that every set can be well-ordered. Finally, he rejected as the source of Burali-Forti's paradox the principle that every well-ordered set has an ordinal and a cardinal number [Hobson 1905, 170-185].

Hobson's critique provoked lengthy articles to the London Mathematical Society by Hardy, Russell, and Jourdain. On 3 August 1905, Hardy submitted the first of these. In particular, he answered Hobson's objections to the construction in [Hardy 1903] of a set of real numbers with power aleph-one. Although Hardy agreed with Hobson that Zermelo's proof should be rejected, he nevertheless accepted Zermelo's *Axiom of Choice* [Hardy 1906]. In a letter of 5 July to Russell, he had revealed his reason for rejecting the proof: "I suppose that Jourdain's and Zermelo's arguments are not unassailable even if the multiplicative class exists [i.e., the Axiom of Choice is true], depending as they do on Burali-Forti's contradiction." For Hardy had accepted the fallacious view, put forward by Hobson [1905] and Bernstein [1905], that both proofs utilized the paradox of the largest ordinal.

This letter of 5 July 1905 was the last in a series which Hardy and Russell exchanged concerning issues that Hobson had raised, particularly the Axiom of Choice and the paradoxes. On 10 June Hardy had enquired:

> *Do you regard Burali-Forti's argument as unassailable, apart from denying the multiplicative class [Axiom of Choice]? I am half inclined to agree with Schoenflies [1905] that it is at bottom unmeaning, though I can't put it even to myself in a satisfying way. Isn't it possible that your original solution was right (denying that W is well-ordered) ...?*

Two days later Russell responded to this query by stressing the importance of predicative functions, i.e., those propositional functions $\phi(x)$ that do not allow as an argument any propositional function defined in terms of $\phi(x)$:

> *I am sorry I gave the impression that denying the multiplicative class solved Burali-Forti. It is not this, but insistence upon the use of what I call a predicative function, as the defining intension, that gives the solution.... I still uphold the Aleph-series; Burali-Forti is avoided by denying that $\alpha < \beta$ defines a rela-*

tion is extension, for then the whole series of Alephs in order of magnitude does not have a type at all, so that there is no difficulty in denying the maximum ordinal.

During the summer Russell continued to ponder these matters. On 24 September he wrote to Couturat concerning a new discovery:

> *As for my own work, I have recently begun to consider Burali-Forti's contradiction. I found the following generalisation: Let $\phi!x$ be any property (propositional function), which always belongs to $f'u$ (where $f'u$ is any function of the class u) when it belongs to all the terms of u; and suppose that one always has $f'u \sim\epsilon\, u$ under the same conditions. Then consider the class $\hat{x}(\phi!x)$. One obtains*
>
> $$\phi!f'\hat{x}(\phi!x) . f'\hat{x}(\phi!x) \sim\epsilon\, \hat{x}(\phi!x),$$
>
> *which is a contradiction [(A)].... In the case of Burali-Forti, $\phi!x . = x$ is an ordinal number, and $f'u . = .$ successor of the class u.... What is interesting is that all the contradictions and paradoxes appear to be particular cases of (A).* [Appendix 7]

Shortly afterward, he used this generalized Burali-Forti paradox in his response to Hobson [Russell 1906, 34-36].

Russell submitted his long and insightful reply to Hobson's critique on 24 November 1905. Above all, he underlined the distinction between the problem of inconsistent classes and that of the Axiom of Choice. In the first case he denied that every propositional function $\phi(x)$ determined a class $\{x:\phi(x)\}$ and in the second he doubted that the Axiom of Choice was true in general. To resolve the problem of inconsistent classes, he discussed three sorts of theories: zigzag theories, *theories of limitation of size*, and no-classes theories. The first sort permitted many propositional functions but prohibited those which were in some sense too complex. Russell noted that it had already been used in *The Principles* [sections 103 and 104]. Such a theory would assert that the complement of a class was a class; it would provide a largest cardinal but no largest ordinal. However, he added, the axioms for a zigzag theory were quite complicated and lacked intrinsic plausibility [Russell 1906, 40].

Next Russell considered theories of limitation of size, such as Jourdain's. [Indeed Cantor's correspondence with Dedekind in 1899, which long remained unpublished, as well as Zermelo's later axiomatization [1908], envisioned some such limitation.) Such a theory would deny that the complement of a class was a class. Furthermore, it would allow no greatest ordinal or cardinal, and no class of all classes or of all ordinals. Yet Russell remained dubious about these theories because no one knew how far the series of ordinals extended. Consequently he turned to a no-classes theory in which "classes and relations are banished altogether" [Russell 1906, 45]. This theory used substitutions in order to treat any propositional function as a mere abbreviation for a statement about one or more of its values. Despite its technical complications, it was the least likely of the three sorts of theories to generate paradoxes. This security highly recommended a no-classes theory to Russell, who did not know whether one could deduce from it even that ω_1 and aleph-one existed.

Jourdain altered his perspective a little, but only a little, in his reply to Hobson for the London Mathematical Society [Jourdain 1906]. He now distinguished W, the set of all ordinal numbers ordered by magnitude, from $\overline{W}$, a set such that every well-ordered set was similar to it or to one of its segments. While W could be extended, $\overline{W}$ could not. Indeed, Jourdain insisted, W was similar to a segment of $\overline{W}$. Finally, he ceased using the term inconsistent set, since he now denied only that W had an order type or cardinal number, not that it was a legitimate set [Jourdain 1906, 267-269].

In part, Jourdain's shift in perspective had been stimulated by Bernstein's recent article [1905], which contained the first discussion of Burali-Forti's paradox to be published in Germany. For Jourdain [1906] objected to the solution that Bernstein had offered to the paradox of the largest ordinal.

Despite working closely with Cantor, Bernstein had not mentioned a single paradox in his doctoral thesis, which was concerned in good part with efforts to systematize set theory [Bernstein 1901]. In fact, Bernstein learned of the paradox of the largest ordinal from Jourdain's article [1904]. When Bernstein first discussed this paradox in 1905, he cited Jourdain as the source for the statement that Cantor had discovered Burali-Forti's paradox in 1895. However, Jourdain had stated merely that during 1895 Cantor found a proof that every set can be well-ordered. In this fashion Bernstein's misinterpretation helped create the myth that Cantor believed his proof to engender a paradox.

In Bernstein's attempt [1905] to refute both Jourdain's and Zermelo's proofs that every set can be well-ordered, Burali-Forti's paradox served as the principal tool. Bernstein did not admit that the set W of all ordinals (or, as he preferred to define it, the set of all order-types of segments of well-ordered sets) was contradictory. Rather he considered the paradox to reside in the assumption that there exists a set $W \cup \{b\}$ in which *b* follows all the elements of *W*. Such a set $W \cup \{b\}$ was not objectionable, Bernstein insisted, as long as one did not permit an order to be imposed on it. Here he rejected Cantor's principle that for every ordinal β there is a next ordinal β + 1. On the other hand, he stated certain "positive" properties of *W*: It was well-ordered but possessed a cardinal number that was not an aleph. While $\bar{W}$ was the largest ordinal, there existed larger cardinals such as that of the power set *Z* of *W*. Moreover, he believed *Z* to provide the simplest example of a set that cannot be well-ordered [Bernstein 1905, 187-192].

In conclusion, Bernstein dismissed Jourdain's proof that every set can be well-ordered, because it relied on the assumption that *W* is inconsistent. Likewise he rejected Zermelo's proof since it used the assumption that every well-ordered set can be extended to a more inclusive well-ordered set [Bernstein 1905, 193]. Arthur Schoenflies' article [1905], which appeared immediately before Bernstein's in *Mathematische Annalen*, essentially agreed with this critique.

Burali-Forti's paradox entered France by a somewhat different path. Early in 1901, Couturat was aware of a difficulty connected with Burali-Forti's article [1897a]. Yet Couturat was inclined to believe simply that some error had crept into Burali-Forti's argument that trichotomy did not hold for ordinals. As a result, Burali-Forti's paradox did not become a matter for public discussion among French mathematicians until 1905. This discussion was stimulated by Bernstein's article [1905] and took place in an exchange of letters (soon published by the *Société Mathématique de France*) between Hadamard, Emile Borel, and others [Hadamard 1905]. Replying to Borel's criticisms of Zermelo's proof, Hadamard analyzed Bernstein's treatment of the paradox of the largest ordinal. In particular, Hadamard refused to grant that one could accept *W* as a set while not permitting any element to come after it. Since the adjunction of new elements was a matter of convention, he insisted that one was always free to add them. "The solution," he wrote concerning Burali-Forti's paradox, "is different. It is the very existence of the set *W* that generates a contradiction.... One has the right to form a set only with previously existing objects, and it is easily seen that the definition of *W* presupposes the opposite" [Hadamard 1905, 271].

Meanwhile Couturat sent to Russell on 16 April 1905 some comments concerning the book that he was writing to spread the ideas in *The Principles*:

> *I am happy to learn that you have resolved the "contradiction" [Russell's paradox]. I will not speak of it in my book [1905] so as not to confuse the reader and make him sceptical.... I fear that an opponent of*

> *Logistic, without bothering to study it seriously, will seize on the "contradiction" in order to discredit this discipline. I admit that I would be rather embarassed to respond to such an objection, which would be dangerous in the eyes of the uninitiated: "How is it, they might ask, that these people claim to reform classical logic and to give it rigorous foundations, while they concede that there is a contradiction in their own principles!" This objection does not seem very serious to me, but that is partly due to an act of faith in reason, which is annoying. What a failure for rationalism if this contradiction was fundamental and incurable! And what revenge for Kant, whose antinomies are child's play by comparison!* [Appendix 6]

In fact, soon after this letter was written an opponent of symbolic logic, Henri Poincaré, did precisely what Couturat had feared, but using Burali-Forti's paradox rather than Russell's.

Poincaré attacked set theory and symbolic logic in a series of articles which appeared in the *Revue de Métaphysique et de Morale* and which analyzed works by Hilbert, Russell, Zermelo, and Burali-Forti. Apparently he had learned of Burali-Forti's paradox from Hadamard [Poincaré 1905, 824]. Unlike Jourdain and Bernstein, he gave every indication of having read Burali-Forti's paper of 1897. After ridiculing Burali-Forti's definition of the number one, written in Peano's symbolism, he added sarcastically: "What makes the paper important is that it gives the first example of these antinomies which one encounters in studying transfinite [ordinal] numbers and which have been for some years the despair of mathematicians" [Poincaré 1905, 822-823]. Here he was influenced by Russell in that he ignored Burali-Forti's distinction between perfectly ordered and well-ordered sets, and in that he generated the paradox by juxtaposing Cantor's and Burali-Forti's articles.

Incensed by Poincaré's cavalier treatment of Russell's work, Couturat composed a lengthy reply for the same journal. Among many other matters, he discussed the paradox of the largest ordinal. Concerning this forthcoming reply to Poincaré, he wrote to Russell on 17 December 1905:

> *The question that embarasses me the most is naturally the mathematical one, namely, the apparent contradiction between Cantor and Burali-Forti. Peano tells me that these two authors have corresponded on this matter, and recognize that they are not speaking about the same ordinal numbers.* [Appendix 8]

Couturat's reply [1906] insisted that this contradiction had nothing to do with the Peanoesque symbolism in which Burali-Forti had first expressed it, but rather that it concerned the logic of classes--a logic which went back to Aristotle. He then quoted from a letter which Burali-Forti had recently sent him about the paradox:

> *The answer to the critique which Poincaré directed at me is already found in my article [1897a,b]. I recall here in a few words all, I believe, that is needed to pose the question clearly.*
>
> *From the collection of Cantor's well-ordered classes I extract a special collection, that of the perfectly ordered classes. A perfectly ordered class is also a well-ordered class; therefore I may consider, along with Cantor, the order-types NO of perfectly ordered classes. Each NO is one of Cantor's ordinal numbers. (But some of the latter may not be contained in my class NO.) Assuming Cantor's theorem, "if a,b are ordinal numbers, one must always have a < b, or a = b, or a > b," I prove that: "The NO, ordered by magnitude, form a perfectly ordered class." (It is important to note: the NO, and not "Cantor's ordinal numbers.")* [Couturat 1906, 228-229]

In 1906, Couturat reported, Burali-Forti still believed that the apparent contradiction between his result and that of Cantor was due to the difference between well-ordered and perfectly ordered classes. Surprisingly, he informed Couturat that there was a printer's error in his article [Burali-Forti 1897b], interchanging instances of "well-ordered" and "perfectly ordered." Thus in 1897, as in 1906, Burali-Forti mistakenly held that every perfectly ordered set is well-ordered, but not conversely. In point of fact, every well-ordered set is perfectly ordered, but some perfectly ordered sets are not well-ordered--in particular all those perfectly ordered sets containing subsets of order-type $\omega(1+{}^*\omega)$. An example of a set that is perfectly ordered but not well-ordered was first given by W. H. Young and G. C. Young [1906, 152].

Although no one paid much notice, in 1906 Poincaré clarified the matter in an article responding to Couturat's:

> *After my article [1905] appeared, Burali-Forti wrote to Couturat. There is no contradiction he claimed, because Cantor's result applies to well-ordered sets and mine to perfectly ordered sets....*
>
> *Couturat quotes Burali-Forti as saying: A perfectly ordered class is also well-ordered, but the*

> *converse is not true. Surely he meant: A well-ordered class is also perfectly ordered, but the converse is not true....*
>
> *Even after this correction, Burali-Forti's explanation is not satisfactory. His reasoning, indeed, is easily applied to well-ordered sets and to Cantor's ordinal numbers. In particular, it is easy to show that the sequence of all Cantor's ordinals forms a well-ordered set.* [Poincaré 1906, 304]

Here Poincaré was correct in two respects. He illuminated Burali-Forti's misunderstanding about the relation between perfectly ordered and well-ordered sets. Moreover, he gave a very lucid account of how Burali-Forti's article should be viewed: While the article did not contain a paradox as it was published, it could be *transformed* into a paradox by mimicking the same argument for well-ordered sets and then juxtaposing Cantor's trichotomy law for ordinals.

Burali-Forti's confusion in his article of 1897 had not escaped Cantor. On 9 March 1907 he wrote to Grace Young to congratulate both her and her husband on their book, *The Theory of Sets of Points* [1906]. Then he added indignantly:

> *Do not fall into the error of those who cast doubt on the reality and consistency of the alephs; these numbers have* the same objective reality *as the finite cardinal numbers known from antiquity. What Schoenflies [1905] calls W is not a "set" in my sense of the word, but an "inconsistent multiplicity." When I wrote the "Grundlagen" [1883b] I already saw this clearly, as is evident from the remarks (1) and (2) in its conclusion, where I referred to W as the "absolutely infinite number-sequence."*
>
> *In (1) I said explicitly* that *I designate as "sets" only those multiplicities* that *can be conceived as* unities, *i.e.*, as objects.... *What Burali-Forti has put forward is utterly foolish. If you go back to his article in the* Circolo Matematico, *you will remark that* not once has he interpreted the concept of "well-ordered set" correctly.
>
> *Enough* about this *for* today. [Appendix 9]

6. CONCLUSION

The origins of Burali-Forti's paradox are at once more complex, more confused, and more intriguing than had been thought. In large measure the complexity stems from the divergent perspectives of the mathematicians involved. As we have seen, neither Burali-Forti nor Cantor believed that there was a genuine paradox of the largest ordinal, much less that he himself had invented it. In 1907 Cantor insisted that during 1883 he had already understood the concept of set in a way that gave rise to no such ordinal. Burali-Forti continued to insist, though incorrectly, that there was no paradox of the largest ordinal because his result applied to perfectly ordered classes, not to well-ordered classes.

Burali-Forti's initial confusion concerning the definition of well-ordered set lay at the root of his refusal to create a paradox. Although he appeared in October 1897 to recognize that every well-ordered set is perfectly ordered but not conversely, in fact he did not do so. It was Poincaré who clarified the matter in 1906.

Still more intriguing is the way in which Russell inadvertently created the paradox of the largest ordinal. His philosophical predilection to seek paradoxes in set theory was already evident by 1896. The first paradox that he created was the paradox of the largest cardinal, which began to take form in 1899 but only became definitive in 1901. From it emerged the paradox that bears his name. Learning of Burali-Forti's work from Couturat, Russell formulated the paradox of the largest ordinal in late 1902 or early 1903 by juxtaposing Burali-Forti's and Cantor's articles of 1897, and by ignoring the difference between well-ordered and perfectly ordered classes.

Yet even in *The Principles*, Russell's understanding of the paradox of the largest ordinal remained fluid. In particular, he did not fully recognize the significance of this paradox until Jourdain emphasized it in his own work. During 1905 the paradox spread to Germany through Bernstein and then to France. In the minds of mathematicians this paradox soon became entangled with Jourdain's and Zermelo's proofs that every set can be well-ordered, as well as with Zermelo's Axiom of Choice.

For later generations, Russell's creation of the paradox of the largest ordinal, and the early attempts at its resolution, blurred what Burali-Forti had actually done. Ironically, the name that Russell gave to his own paradox of the largest ordinal has endured: Burali-Forti's paradox.

APPENDIXES

The Couturat-Russell correspondence, excerpts from which are reproduced below, is kept in the Bibliothèque of the town La Chaux-de-Fonds in Switzerland. The Russell Archives contain photocopies of these letters. Finally, the letter from Cantor to Grace Young can be found in the Archives of the University of Liverpool.

Appendix 1: Couturat to Russell, 3 January 1901

... Les erreurs que vous me signalez dans Cantor me paraissent fort intéressantes, mais je suis bien loin de ce sujet, et, plongé dans Leibniz, je n'ai pas le loisir de l'étudier. L'infini prête si aisément aux paralogismes! Burali-Forti a prétendu démontrer que, dans les types d'ordre, il est faux d'affirmer:

$$(a = b) \vee (a < b) \vee (a > b)$$

et que par suite ils ne forment pas une classe bien ordonnée. Son raisonnement est plus spécieux que probant.[1] Je me demande si l'on peut considérer la classe de toutes les classes possibles sans une espèce de contradiction.

1. Una questione sui numeri transfiniti, in Rendiconti del Circolo matematico di Palermo (28 mars 1897) (t. XI). Je peux vous prêter l'article, si vous le désirez.

Appendix 2: Russell to Couturat, 17 January 1901

... Je suis d'accord avec Burali-Forti qu'il soit faux d'affirmer pour les types d'ordre:

$$a=b \;.\vee.\; a<b \;.\vee.\; a>b$$

ce qui est plus, je soupçonne qu'on ne puisse affirmer la même chose pour les nombres cardinaux. Les arguments de Cantor à ce sujet ne sont pas concluants. Je vous serai très reconnaissant si vous voulez bien me prêter l'article de Burali-Forti, que je n'ai pas vu.—Quant à la classe des classes, je vous accorde une contradiction dans ce concept. L'infini reste toujours contradictoire, et les travaux ainsi que ceux de Cantor n'ont pas résolu le problème philosophique. Car il y a un concept classe et il y a des classes. Donc classe est une classe. Je ne prouve [illegible] essentiel à la théorie de Cantor que toute classe a un nombre cardinal.

Appendix 3: Couturat to Russell, 27 January 1901

... La classe classe est-elle déterminée, fermée en quelque sorte, de manière à posséder un nombre cardinal?

Appendix 4: Couturat to Russell, 21 July 1901

... Il est très profond sous sa forme populaire, et n'a beaucoup plu. Tous les paradoxes se cachent des vérités frappantes.

Appendix 5: Russell to Couturat, 29 September 1902

... Je suis très occupé par mon livre, qu'on imprime lentement, et par le Vol. II que je prépare en collaboration avec Whitehead. Je ne sais que faire d'une classe de contradictions, dont voici la plus simple

APPENDIX 5 [illegible]

J'ai essayé maintes solutions sans succès. On trouve des contradictions de ce genre en appliquant à la classe de tous les individus, ou de toutes les propositions, ou de toutes les relations, la preuve que donne Cantor qu'il n'y a pas de nombre cardinale maximum. Quand on a commencé à imprimer mon livre, j'ai cru pouvoir éviter ces contradictions, mais je vois à présent que je me trompais, ce qui diminue de beaucoup la valeur de mon livre.

Appendix 6: Couturat to Russell, 18 April 1904

... Je suis heureux d'apprendre que vous êtes venu à bout de la "Contradiction"; je n'en parle pas dans mon livre, pour ne pas embrouiller le lecteur et le rendre sceptique.... Je crains, entre nous, qu'un des adversaires de la Logistique, sans se donner la peine d'étudier sérieusement cette doctrine, ne s'empare de la "contradiction" pour la discréditer. J'avoue que je serais assez embarrassé pour répondre à une telle objection, qui serait dangereuse aux yeux du public profane. "Comment, dirait-on, voici ces gens qui prétendent réformer la logique classique et lui donner des fondements rigoureux, et ils aboutissent une contradiction au sein même de leurs principes !" Cette objection sophistique me paraît peu sérieuse, mais c'est un des [illegible] par un acte de foi dans la raison, ce qui est ennuyeux. Quel échec pour le rationalisme, si cette contradiction était fondamentale et incurable! Et quelle revanche pour Kant, dont les antinomies ne sont que jeu d'enfants à côté!

Appendix 7: Russell to Couturat, 24 September 1905

... Quant à mes travaux personnels, je me suis mis dernièrement à considérer la contradiction de Burali-Forti. J'ai trouvé la généralisation que voici: Soit ϕx une propriété (fonction propositionnelle) quelconque, qui appartient toujours à $f'u$ quand u est une fonction quelconque de la classe et quand elle appartient à tous les termes de u, et supposez qu'on ait toujours $f'u \sim \epsilon\, u$ dans les mêmes circonstances. Qu'on considère alors la classe $\hat{w}(\phi w)$. On trouvera

$$f'\hat{w}(\phi w) \,.\, \sim \{f'\hat{w}(\phi w) \,\epsilon\, \hat{w}(\phi w)\}\,,$$

ce qui est une contradiction (A).... Dans le cas de Burali-Forti, ϕx. = x est un nombre ordinal, et $f'u$ = successeur de la classe u .... Ce qui est intéressant c'est que toutes les contradictions paraissent être des cas particuliers de (A).

Appendix 8: Couturat to Russell, 17 December 1905

... La question qui m'embarrasse le plus est naturellement la question mathématique, à savoir la contradiction apparente entre Cantor et Burali-Forti. M. Peano me dit que les deux auteurs ont correspondu à ce sujet, et reconnaissent qu'ils ne parlent pas des mêmes nombres ordinaux.

Appendix 9: Cantor to G. H. Young, 9 March 1907

... Lassen Sie sich nicht von jenen irre machen, die an der Realität und Widerspruchslosigkeit der Alefzahlen glauben zweifeln zu sollen; diese Zahlen haben dieselbe feste Möglichkeit wie die von Alters her bekannten endlichen Cardinalzahlen. Was Herr Schoenflies W nennt, ist keine "Menge" in der von mir gewählten Sinne des Wortes, sondern eine "inconsistente Vielheit." Schon als ich die "Grundlagen" schriebe, habe ich dies klar gesehen, wie aus den Anmerkungen (1) und (2) am Schlusse hervorgeht, wo ich W die "absolut unendliche Zahl-folge" nenne.

In [1] sage ich ausdrücklich, dass ich nur solche Vielheiten "Mengen" nenne, die ohne Widerspruch als Einheiten, d.h. als Dinge gedacht werden können.... Was Burali-Forti vorgebracht hat, ist höchst thöricht. Wenn Sie auf seine Abh. im Circolo Matem. zurückgehen, werden Sie bemerken, dass er nicht einmal den Begriff der "wohlgeordneten Menge" richtig aufgefasst hatte.

Doch genug hiervon für heute.

ACKNOWLEDGMENTS

Our research was greatly facilitated by Kenneth Blackwell and Carl Spadoni at the Bertrand Russell Archives (McMaster University, Hamilton, Canada). We gratefully acknowledge permission from the Archives and the Russell Estate to quote from Russell's unpublished letters and manuscripts, as well as permission from the Masters and Fellows of Trinity College (Cambridge, England) to quote from Hardy's letters. We are grateful to Pierre Hirsch (Bibliotèque de la ville, La Chaux-de-Fonds, Switzerland) for permission to quote from the Couturat-Russell correspondence. We also thank Mrs. R. C. H. Tanner, who gave us permission to quote from Cantor's letter. Finally, this article has benefited from the constructive criticism of William Aspray, Joseph Dauben, Ivor Grattan-Guinness, Esther Phillips, Alisdair Urquhart and those in the I.H.P.S.T. Seminar in the History of Mathematics at the University of Toronto.

NOTES

1. Moore arrived at this conclusion in earlier articles [1978, 308-310; 1980, 102-105]. During 1979 Garciadiego investigated the matter further in two unpublished papers written at the University of Toronto: "The Origin of Burali-Forti's Paradox" and "A New Historical Interpretation of the Paradoxes of Set Theory." The present article incorporates, but proceeds appreciably beyond, the earlier papers of both authors.

2. Vis-à-vis Russell's paradox, Alberto Coffa has adopted a similar viewpoint: "Thus perhaps the impact of the paradox emerged over a period of time, and perhaps their decisions played no less significant a role than extensive discoveries" [1979, 37]. In an article entitled "Are There Paradoxes of the Set of All Sets?" (forthcoming in the *International Journal of Mathematics Education*), Ivor Grattan-Guinness has criticized Coffa's [1979] reconstruction of the origins of Russell's paradox. It should be noted that Thomas Kuhn, who studied how a Gestalt switch takes place when one scientific theory is replaced by another [1970, 111-135], was concerned with issues different from those we are concerned with here.

3. Although Burali-Forti stated condition (c) in this form in 1897, he had employed a stronger condition in 1896: Every element has an immediate successor. Like all of those who followed Peano's terminology in mathematical logic (and this includes Russell), Burali-Forti spoke of classes rather than sets.

4. Russell [1903, 45] noted that Burali-Forti's postulate (1) was false unless one also stipulated that any two classes in A are disjoint. On the other hand, postulate (2) is equivalent to Zermelo's later Axiom of Choice; see Moore [1978, 78-79].

5. This passage, originally in French, can be found in Coffa [1979, 33]. Our translation differs slightly from his. All the subsequent letters between Couturat and Russell that we have translated in the text are previously unpublished and are quoted with permission. The original letters are printed in the appendices. A definitive edition of the Couturat-Russell correspondence, extending from 1897 to 1914, is being prepared by Anne-Françoise Schmid.

6. In his article [1892], Cantor gave a diagonalization argument to show that the set of real numbers is uncountable, a result that he had previously obtained by more specialized means. He went on to claim that there is no largest cardinal, since, for any set M, the set of all functions from M to {0,1} would have a cardinal larger than M. The function that he used, in the case when M is the set of all real numbers, was vital to Russell's eventual formulation of the paradox of the largest cardinal.

7. For a study of the philosophical and mathematical aspects of Russell's unpublished correspondence with Alys, see Spadoni [1978].

8. Quoted in Spadoni [1978, 29-37]. Russell also mentioned to Frege that he had recently sent Peano a letter about the paradox; this letter is now lost (see [Kennedy 1975, 209]).

9. An intriguing study of the changes in *The Principles*, particularly as they relate to Russell's paradox, can be found in Blackwell [unpublished].

10. Russell [1903, 367]. Russell designated the paradox of the largest cardinal as "Cantor's paradox" since it depended in part on Cantor's 1891 argument that there is no largest cardinal.

11. At this time Russell temporarily believed that he had solved his paradox. The Russell Archives possesses a telegram, dated 21 May 1903, in which A. N. Whitehead congratulated Russell on this feat. This letter of 14 October, and the following ones between Hardy and Russell (of which only excerpts have been published), can be found in the Trinity College Library of Cambridge University. Certain excerpts from the letters of 30 June and 1 July 1905 are printed in Grattan-Guinness [1978, 131-133].

12. For a discussion of how either Russell or Cavaillès conflated these two letters, see Grattan-Guinness [1978, 126-131, 136-138].

13. Jourdain [1904] cited *The Principles*, and all other references, by page number--with the sole exception of Burali-Forti's article [1897a]. When citing that article in *The Principles*, Russell had not stated a page reference, since his copy of the article was an offprint (now in the Russell Archives) paginated from 1 to 1[illegible]

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